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Equations of the moduli of Higgs pairs and Infinite Grassmannian

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In this paper the moduli space of Higgs pairs over a fixed smooth projective curve with extra formal data is defined and is endowed with a scheme structure. We introduce a relative version of the Krichever map using a fibration of Sato Grassmannians and show that this map is injective. This, together with the characterization of the points of the image of the Krichever map, allows us to prove that this moduli space is a closed subscheme of the product of the moduli of vector bundles (with formal extra data) and a formal analogue of the Hitchin base. This characterization also provides us with a method for explicitly computing KP-type equations that describe the moduli space of Higgs pairs. Finally, for the case where the spectral cover is totally ramified at a fixed point of the curve, these equations are given in terms of the characteristic coefficients of the Higgs field.

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1. Introduction.

In 1988, Hitchin studied ([5]) the symplectic geometry of the cotangent space $T^*\mathcal{U}_X$ to the moduli space $\mathcal{U}_X$ of vector bundles over a compact Riemann surface X . Its points led him to introduce a new and currently relevant concept: the notion of Higgs pairs. From the point of view of symplectic geometry, the map from $T^*\mathcal{U}_X$ to an affine space of global sections (now called the Hitchin map) turned out to be an algebraically completely integrable Hamiltonian system. This translates in algebraic geometry by saying that the fibers of the Hitchin map are Jacobians of a special curve covering X (the spectral curve). As the author states in [5], one question is left: can we find the Hamiltonian differential equations in some concrete way?

Bearing this question in mind, Li and Mulase partially solved the problem in [7] using techniques of infinite integrable systems. They used the infinite Sato Grassmannian as a space of solutions for the KP system ([15]) and answered the question of how to recover the Hitchin system from the KP system in the case for which the spectral cover is not ramified. In the same period, and with similar tools (Krichever

2 *D. Hernández-Serrano, J. M. Muñoz Porras, F. J. Plaza Martín*

morphism, Sato Grassmannian, ...), Donagi and Markman also studied this topic in [4] with full generality. They demonstrated a certain compatibility for KP flows concerning spectral data coming from Higgs pairs on X . As far as we know, concrete equations describing this moduli space (and allowing ramification in the spectral cover) remain to be given.

The paper is organized as follows. In section 2, we offer a few words about the classical correspondence between Higgs pairs and line bundles over the spectral curve. Section 3 is devoted to the study of the Krichever map for the moduli space of Higgs pairs with extra formal data, $\mathcal{Higgs}_X^\infty$. For this goal, an alternative method to that of [7] and [4] is proposed: the use of a fibration of infinite Sato Grassmannians (over a formal analogue $\mathbb{A}$ of the Hitchin base) and a generalization of the Krichever morphism. We show that the Krichever map is injective (Theorem 3.2) and a characterization of its image is also given (Theorem 3.3), allowing us to state the first main result: $\mathcal{Higgs}_X^\infty$ is a closed subscheme of $\mathcal{U}_X^\infty \times \mathbb{A}$ (Theorem 3.4, where $\mathcal{U}_X^\infty$ is the moduli scheme of vector bundles over X with formal extra data). Section 4 gives the spectral construction for the above topics, defining a relative Grassmannian. Finally, in section 5 the second main result is given: Theorem 5.4 provides concrete KP-type equations describing $\mathcal{Higgs}_X^\infty$ in terms of bilinear identities of Baker-Akhiezer functions and, for the subscheme of $\mathcal{Higgs}_X^\infty$ on which the spectral cover is totally ramified, the equations are explicitly computed in terms of the characteristic coefficients of the Higgs field (Theorem 5.5).

2. Preliminaries.

We shall work over an algebraically closed field k of characteristic zero. Let $\mathcal{U}_X$ denote the moduli stack of rank n vector bundles over a projective, smooth curve X of genus g . Recall that a Higgs pair on X consists of a vector bundle $E \in \mathcal{U}_X$ and a morphism of sheaves of $\mathcal{O}_X$ -modules:

$$\varphi: E \rightarrow E \otimes \omega_X.$$

It is known ([4]) that:

$$T_E^* \mathcal{U}_X \simeq H^0(X, \mathcal{E}nd_X E \otimes \omega_X),$$

and therefore the cotangent space to $\mathcal{U}_X$ parametrizes Higgs pairs on X .

A remarkable fact is the existence of the Hitchin map:

$$\begin{aligned} H: T^* \mathcal{U} &\rightarrow \bigoplus_{i=1}^n H^0(X, \omega_X^{\otimes i}) \\ (E, \varphi) &\mapsto \text{ch}(\varphi) \end{aligned}$$

where:

$$\text{ch}(\varphi) = (\text{Tr}(\varphi), \text{Tr}(\Lambda^2 \varphi), \dots, \text{Tr}(\Lambda^n \varphi))$$

are the characteristic coefficients of φ .

Another relevant issue is the spectral construction, whose importance lies in the fact that, for a generic $a = (a_1, \dots, a_n) \in \oplus_{i=1}^n H^0(X, \omega_X^{\otimes i})$, the points of $H^{-1}(a)$ correspond to line bundles on the so called spectral curve X_a . We briefly describe its definition ([5],[3],[18]). Let us consider the cotangent bundle:

$$T^*X := \text{Spec } \mathcal{S}^\bullet \omega_X^{-1}$$

where $\mathcal{S}^\bullet \omega_X^{-1}$ is the symmetric algebra of ω_X^{-1} . Let $\mathcal{I}_a$ be the sheaf of ideals of $\mathcal{S}^\bullet \omega_X^{-1}$ generated by the image of:

$$\begin{aligned} \omega_X^{-n} &\rightarrow \mathcal{S}^\bullet \omega_X^{-1} = \oplus_{i \geq 0} \omega_X^{\otimes -i} \\ \xi &\mapsto (\xi \otimes (-1)^n a_n, \dots, \xi \otimes -a_1, \xi, 0, 0, \dots). \end{aligned}$$

The quotient $\mathcal{S}^\bullet \omega_X^{-1} / \mathcal{I}_a$ is a sheaf of $\mathcal{O}_X$ -algebras and the spectral curve is defined as the zeros of $\mathcal{I}_a$ in T^*X , that is:

$$X_a := \text{Spec } \mathcal{S}^\bullet \omega_X^{-1} / \mathcal{I}_a.$$

The map:

$$\pi: X_a \rightarrow X$$

is a finite covering of degree n , where n is the rank of E and for generic a X_a is non-singular.

For generic a , the bijection between $\text{Jac}(X_a)$ and $H^{-1}(a)$ is as follows:

We can think of the Higgs field φ as a morphism:

$$\omega_X^{-1} \rightarrow E \otimes E^*,$$

or, equivalently, as a morphism of $\mathcal{O}_X$ -algebras

$$\mathcal{S}^\bullet \omega_X^{-1} \rightarrow E \otimes E^*.$$

The Hamilton-Cayley theorem implies that this morphism factorizes by $\mathcal{O}_{X_a} = \mathcal{S}^\bullet \omega_X^{-1} / \mathcal{I}_a$ if and only if $\text{ch}(\varphi) = a$. Thus, given a point in $H^{-1}(a)$, that is given (E, φ) such that $\text{ch}(\varphi) = a$, E can be thought of as a line bundle L on X_a and $\pi_* L \simeq E$. The converse is analogous.

3. Moduli of Higgs pairs with extra formal data.

We shall assume that the following data are definitively fixed:

- X is a smooth, projective, integral curve over k .
- $x \in X$.
- $t: \widehat{\mathcal{O}}_{X,x} \simeq k[[z]]$ is a formal trivialization of X at x .

Since X is smooth, the dualising sheaf ω_X is a line bundle, and the formal trivialization t induces a formal trivialization of ω_X at x :

$$dt: \widehat{\omega}_{X,x} \simeq k[[z]]dz \tag{3.1}$$

4 *D. Hernández-Serrano, J. M. Muñoz Porras, F. J. Plaza Martín*

(if no confusion arises, we shall forget about the generator dz). We wish to consider a moduli problem parametrising:

- A rank n vector bundle on X , E .
- A Higgs field $\varphi: E \rightarrow E \otimes \omega_X$.
- A formal trivialization ϕ of E on x , i.e. $\phi: \widehat{E}_x \simeq \widehat{\mathcal{O}}_{X,x}^{\oplus n}$

We shall see that a certain compatibility between ϕ and φ must be imposed.

3.1. Definition and representability.

Let us denote by $\widehat{\varphi}$ the completion at x of the Higgs field φ . Given the formal trivialization ϕ , the coefficients a_i of the characteristic polynomial $p_{\widehat{\varphi}}(T)$ of $\widehat{\varphi}$ lie on $k[[z]]$.

Definition 3.1. We define the functor $\mathcal{Higgs}_X^\infty$ as the sheafication of:

$$\begin{aligned} \mathcal{C}_{k-sch} &\rightsquigarrow \mathcal{C}_{sets} \\ S &\mapsto \{(E, \phi, \varphi)\} / \sim, \end{aligned}$$

where

- (1) E is a rank n vector bundle over $X \times S$.
- (2) $\varphi: E \rightarrow E \otimes \omega_{X \times S/S}$ is a relative Higgs field; that is, a morphism of sheaves of $\mathcal{O}_{X \times S}$ -modules.
- (3) ϕ is a formal trivialization of E along $x \times S$,

$$\phi: \widehat{E}_{x \times S} \simeq \widehat{\mathcal{O}}_{x \times S}^{\oplus n},$$

such that there exists a matrix:

$$M = \begin{pmatrix} 0 & \cdot & \cdot & 0 & (-1)^{n+1}a_n \\ 1 & 0 & \cdot & \cdot & (-1)^n a_{n-1} \\ 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & 0 & -a_2 \\ 0 & \cdot & \cdot & 0 & 1 & a_1 \end{pmatrix},$$

(where $(a_1, \dots, a_n) \in \mathcal{O}_S[[z]]^n$) making the following diagram commutative:

$$\begin{array}{ccc} \widehat{E}_{x \times S} & \xrightarrow{\widehat{\varphi}} & \widehat{E}_{x \times S} \otimes \widehat{\omega}_{x \times S} \\ \phi \downarrow \wr & & \wr \downarrow \phi \otimes dt \\ \mathcal{O}_S[[z]]^n & \xrightarrow{M} & \mathcal{O}_S[[z]]^n \end{array}$$

- (4) Two triples (E, φ, ϕ) and (E', φ', ϕ') are said to be equivalent whenever there exists an isomorphism of vector bundles $f: E \simeq E'$ compatible with all data. That is:

- f is compatible with ϕ and ϕ' , and hence the following diagram is commutative:

$$\begin{array}{ccc} \widehat{E}_{x \times S} & \xrightarrow[\sim]{\widehat{f}} & \widehat{E}'_{x \times S} \\ & \searrow \phi \quad \swarrow \phi' & \\ & \mathcal{O}_S[[z]]^n & \end{array}.$$

- f is compatible with φ and φ' ; that is, the diagram:

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E \otimes \omega \\ f \downarrow \wr & & \downarrow \wr f \otimes \text{Id} \\ E' & \xrightarrow{\varphi'} & E' \otimes \omega \end{array}$$

is commutative.

Remark 3.1. Given $(E, \phi, \varphi) \in \mathcal{Higgs}_X^\infty(S)$, the elements $(a_1, \dots, a_n)$ of the third condition are necessarily the characteristic coefficients of $\widehat{\varphi}$.

Remark 3.2. Whenever two S -valued points (E, φ, ϕ) and (E', φ', ϕ') are equivalent, the characteristic coefficients of both $\widehat{\varphi}$ and $\widehat{\varphi}'$ are the same. This implies the commutativity of the following diagram:

$$\begin{array}{ccccc} E & \xrightarrow{\varphi} & & E \otimes \omega & \\ & \searrow & & \swarrow & \\ & \mathcal{O}_S[[z]]^n & \xrightarrow{M} & \mathcal{O}_S[[z]]^n & \\ & \swarrow & & \searrow & \\ E' & \xrightarrow{\varphi'} & & E' \otimes \omega & \\ & \wr \downarrow & & \downarrow \wr & \end{array}$$

Let us denote by $\mathcal{U}_X^\infty$ the moduli space of vector bundles over X with formal trivialization (recall that this moduli space can be obtained as a limit of the moduli of vector bundles with level structure, [17], and in [1] it is shown to be a fine moduli space). It is easy to see that for every vector bundle E on X we can choose a formal trivialization at a fixed point $x \in X$, $\phi: \widehat{E}_x \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}^n$. That is, we can choose a point in $\mathcal{U}_X^\infty$.

Let $\mathcal{Higgs}_X$ denote the stack of Higgs pairs over X .

Proposition 3.1. *Let S be a k -scheme. Given $(E, \varphi) \in \mathcal{Higgs}_X(S)$, there exists a formal trivialization of E along $x \times S$ satisfying the compatibility condition of Definition 3.1.*

Proof. Let (E, φ) be a point in $\mathcal{Higgs}_X(S)$ and let us choose a formal trivialization ϕ of E along $x \times S$. Setting $(a_1, \dots, a_n) \in \mathcal{O}_S[[z]]^n$ to be the characteristic coefficients

6 *D. Hernández-Serrano, J. M. Muñoz Porras, F. J. Plaza Martín*

of the completion $\widehat{\varphi}$ of the Higgs field along $x \times S$, one can easily show that ϕ is compatible with $\widehat{\varphi}$ in the sense of Definition 3.1. $\square$

Theorem 3.1. *The functor $\mathcal{Higgs}_X^\infty$ is representable by a k -scheme.*

Proof. Since $\mathcal{U}_X^\infty$ is a fine moduli space, there exists a universal object $(\mathcal{E}, \Phi)$, where $\mathcal{E}$ is a rank n vector bundle on $X \times \mathcal{U}_X^\infty$ and Φ is a formal trivialization of $\mathcal{E}$ along $\{x\} \times \mathcal{U}_X^\infty$.

Let $\mathbb{V}$ be the fiber bundle associated with the following vector bundle on $X \times \mathcal{U}_X^\infty$:

$$\mathcal{H}om_{X \times \mathcal{U}_X^\infty}(\mathcal{E}, \mathcal{E} \otimes \omega),$$

where ω is the pullback of ω_X to $X \times \mathcal{U}_X^\infty$.

One has that a point of $\mathbb{V}$ with values in a k -scheme S , $g: S \rightarrow \mathbb{V}$, corresponds to a triple (E_S, φ_S, ϕ_S) , where (E_S, ϕ_S) corresponds to the S -valued point of $\mathcal{U}_X^\infty$ given by the composition:

$$S \rightarrow \mathbb{V} \rightarrow X \times \mathcal{U}_X^\infty \xrightarrow{p_2} \mathcal{U}_X^\infty$$

(p_2 being the projection on the second factor) and φ_S is the S -valued point given by g .

By Definition 3.1, the set of S -valued points of $\mathcal{Higgs}_X^\infty$ consists of those S -valued points of $\mathbb{V}$, (E_S, φ_S, ϕ_S) , such that the composition:

$$\mathcal{O}_S[[z]]^n \xrightarrow{\phi_S^{-1}} \widehat{E}_{S,x \times S} \xrightarrow{\widehat{\varphi}_S} \widehat{E}_{S,x \times S} \otimes \widehat{\omega}_{x \times S} \xrightarrow{\phi_S \times dt_S} \mathcal{O}_S[[z]]^n$$

is defined by a matrix of the type:

$$\begin{pmatrix} 0 & \cdot & \cdot & 0 & (-1)^{n+1}a_n \\ 1 & 0 & \cdot & \cdot & (-1)^n a_{n-1} \\ 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & 0 & -a_2 \\ 0 & \cdot & \cdot & 0 & 1 & a_1 \end{pmatrix}$$

for arbitrary $(a_1, \dots, a_n) \in \mathcal{O}_S[[z]]^n$. One can check that this defines a closed subscheme of $\mathbb{V}$. $\square$

3.2. Formal Hitchin map.

We have already stated that the coefficients a_i of the characteristic polynomial $p_{\widehat{\varphi}}(T)$ of $\widehat{\varphi}$ lie on $k[[z]]$. Let us now introduce a k -scheme parametrising these coefficients.

Let $\mathbb{A}^\infty$ be the infinite dimensional affine scheme over k defined in [2] :

$$\mathbb{A}^\infty := \operatorname{Spec} \varinjlim_l k[y_0, \dots, y_l] = \operatorname{Spec} k[y_0, y_1, \dots]$$

Let R be a k -algebra. The correspondence:

$$s_0 + s_1 z + \cdots \in R[[z]] \leftrightarrow (s_0, s_1, \dots) \in \mathbb{A}^\infty(R)$$

implies that the functor that associates the ring of formal power series $R[[z]]$ with each k -algebra, R , is representable by $\mathbb{A}^\infty$. From this fact we obtain the following:

Proposition 3.2. *The functor over the category of k -schemes defined by:*

$$S \rightsquigarrow \oplus_{i=1}^n \mathbf{H}^0(\widehat{X}_x \times S, \widehat{\omega}_{x \times S}^{\otimes i})$$

is representable by the k -scheme $\mathbb{A} = (\mathbb{A}^\infty)^n$.

Remark 3.3. We shall think of the points of $\mathbb{A}$ with values in S as polynomials:

$$T^n - a_1 T^{n-1} + \cdots + (-1)^n a_n,$$

where $a_i \in \mathcal{O}_S[[z]]$ via the identification $\widehat{\omega}_{x \times S}^{\otimes i} \simeq \mathcal{O}_S[[z]]$ induced by t .

Definition 3.2. The *formal Hitchin map* is the morphism of schemes:

$$\mathcal{H}_\infty: \mathcal{Higgs}_X^\infty \rightarrow \mathbb{A}$$

which associates the characteristic coefficients:

$$(a_1, \dots, a_n) \in \mathbb{A}(S) \quad a_i = \text{Tr}(\Lambda^i \widehat{\varphi})$$

of the formal Higgs field $\widehat{\varphi}$ with each triple $(E, \phi, \varphi) \in \mathcal{Higgs}_X^\infty(S)$

3.3. Krichever map.

Let us set $V = k((z))^n$ and $V^+ = k[[z]]^n$. Recall that the infinite Grassmannian $\text{Gr}(V)$ associated with the couple (V, V^+) is an infinite dimensional k -scheme whose rational points are k -vector subspaces W of V such that $W \cap V^+$ and $V/W + V^+$ are finite dimensional (see Definition 2.9 of [2]).

Recall also ([1], [10]) that the Krichever map

$$\mathcal{U}_X^\infty \rightarrow \text{Gr}(V)$$

is defined by sending (E, ϕ) to $(t \circ \phi) \varinjlim_m \pi_* E(m)$ and for rational points it maps (E, ϕ) to $(t \circ \phi) \mathbf{H}^0(X - x, E)$.

Definition 3.3. Let (E, ϕ, φ) be a point of $\mathcal{Higgs}_X^\infty(S)$. The Krichever map for the functor $\mathcal{Higgs}_X^\infty$ is defined by:

$$\begin{aligned} \mathcal{Higgs}_X^\infty(S) &\rightarrow \text{Gr}(k((z))^n)(S) \times \mathbb{A}(S) \\ (E, \phi, \varphi) &\mapsto ((t \circ \phi) \varinjlim_m \pi_* E(m), p_{\widehat{\varphi}}(T)), \end{aligned}$$

where $\pi: X \times S \rightarrow S$ and $p_{\widehat{\varphi}}(T)$ is the characteristic polynomial of the formal Higgs field $\widehat{\varphi}$.

8 *D. Hernández-Serrano, J. M. Muñoz Porras, F. J. Plaza Martín*

Note that for rational points, the Krichever map sends:

$$(E, \phi, \varphi) \mapsto ((t \circ \phi) H^0(X - x, E), p_{\widehat{\varphi}}(T))$$

Theorem 3.2. *The Krichever map is injective.*

Proof. Let (E, ϕ, φ) and (E', ϕ', φ') be two points in $\mathcal{Higgs}_X^\infty$ with values in the k -scheme S and let us assume that:

$$(t \circ \phi) \varinjlim_m \pi_* E(m) = (t \circ \phi') \varinjlim_m \pi_* E'(m)$$

$$p_{\widehat{\varphi}}(T) = p_{\widehat{\varphi}'}(T).$$

Since the Krichever map for $\mathcal{U}_X^\infty$ is injective (see [1],[10]), there exists an isomorphism $E \simeq E'$ compatible with the formal trivializations ϕ and ϕ' . Because both families satisfy the third condition of Definition 3.1, it follows that the diagram:

$$\begin{array}{ccccc} E & \xrightarrow{\quad \varphi \quad} & E \otimes \omega & & \\ & \searrow & \downarrow \wr & \nearrow & \\ & \mathcal{O}_S[[z]]^n & \xrightarrow{M} & \mathcal{O}_S[[z]]^n & \\ & \nearrow & \downarrow \wr & \searrow & \\ E' & \xrightarrow{\quad \varphi' \quad} & E' \otimes \omega & & \end{array}$$

is commutative, and therefore $(E, \phi, \varphi) \sim (E', \phi', \varphi')$ (see remark 3.2). $\square$

Let us consider the natural map:

$$V \otimes_k k((z)) \rightarrow V = V \otimes_{k((z))} k((z)).$$

If W and U are vector subspaces of V and $k((z))$ respectively, we denote by $W \cdot U$ the image of $W \otimes_k U \rightarrow V$.

Proposition 3.3. *Let L be a line bundle on X and let ϕ_L be a formal trivialization of L at x . Let us consider $(E, \phi_E) \in \mathcal{U}_X^\infty(k)$, and let:*

$$W = H^0(X - x, E) \in \text{Gr}(V)(k), \quad U = H^0(X - x, L) \in \text{Gr}(k((z)))(k)$$

be the points defined via the Krichever map. We thus have:

$$W \cdot U = H^0(X - x, E \otimes_{\mathcal{O}_X} L) \in \text{Gr}(V)(k).$$

Proof. This follows from the surjectivity of:

$$H^0(X - x, E) \otimes_k H^0(X - x, L) \rightarrow H^0(X - x, E \otimes_{\mathcal{O}_X} L). \quad \square$$

One can easily generalize the above result for points with values in any k -scheme S . This implies that we can define the following morphism of schemes:

$$\begin{aligned} h_U: \mathcal{U}_X^\infty &\rightarrow \mathcal{U}_X^\infty \\ W &\mapsto W \cdot U \end{aligned}$$

Remark 3.4. Note that in general we do not have a well-defined scheme morphism:

$$\begin{aligned} \mathrm{Gr}(V) \times \mathrm{Gr}(k((z))) &\rightarrow \mathrm{Gr}(V) \\ (W, U) &\mapsto W \cdot U \end{aligned}$$

We now wish to characterize the image of the Krichever map. It is known ([1, 10]) that the image of the Krichever map:

$$\mathcal{U}_X^\infty \hookrightarrow \mathrm{Gr}(V)$$

consists of those points $W \in \mathrm{Gr}(V)$ such that $A \cdot W \subseteq W$ (where $A = H^0(X - x, \mathcal{O}_X)$, see [13]. Notice also that $A \cdot W = W \cdot A$). Since:

$$\mathcal{Higgs}_X^\infty \hookrightarrow \mathrm{Gr}(V) \times \mathbb{A}$$

takes values in $\mathcal{U}_X^\infty \times \mathbb{A}$, we are interested in studying the image of:

$$\mathcal{Higgs}_X^\infty \hookrightarrow \mathcal{U}_X^\infty \times \mathbb{A}.$$

Definition 3.4. Let $W \in \mathrm{Gr}(V)(S)$ be a point with values in S . A sub- $\mathcal{O}_S$ -algebra A of $\mathcal{O}_S((z))$ is said to *stabilize* W if it satisfies the following two conditions:

- (1) $\mathcal{O}_S((z))/A$ is flat over S .
- (2) $A \cdot W \subseteq W$.

The *stabilizer algebra* of W , A_W , is the maximum subalgebra of $\mathcal{O}_S((z))$ that stabilizes W .

If $S = \mathrm{Spec} k$ is the spectrum of a field, flatness is fulfilled and the stabilizer algebra coincides with that of [10]:

$$A_W = \{f \in k((z)) \mid f \cdot W \subseteq W\}.$$

Proposition 3.4. Let S be a k -scheme, $A_S := H^0(X - x, \mathcal{O}_X) \otimes_k S$ and let T be an $\mathcal{O}_S((z))$ -linear automorphism of $\mathcal{O}_S((z))^n$. Given $W \in \mathcal{U}_X^\infty(S)$, it holds that:

- (1) $T(W) \in \mathrm{Gr}(V)(S)$.
- (2) $T(W)$ is an A_S -module.
- (3) $A_{T(W)} = A_W = A_S \in \mathrm{Gr}(k((z)))(S)$.

In particular, $T(W) \in \mathcal{U}_X^\infty(S)$.

Proof. This follows from the $\mathcal{O}_S((z))$ -linearity of T and the properties of the stabilizer algebra, A_W , of W . $\square$

We can now define another endomorphism of $\mathcal{U}_X^\infty$:

$$\begin{aligned} T: \mathcal{U}_X^\infty &\rightarrow \mathcal{U}_X^\infty \\ W &\mapsto T(W). \end{aligned}$$

Theorem 3.3. *Let Ω be the point in $\mathrm{Gr}(k((z)))(S)$ defined by the relative canonical sheaf $\omega_{X \times S/S}$, $(W, p(T)) \in (\mathcal{U}_X^\infty \times \mathbb{A})(S)$ and T the $\mathcal{O}_S((z))$ -linear automorphism of $\mathcal{O}_S((z))^n$ given by:*

$$T = \begin{pmatrix} 0 & \cdot & \cdot & 0 & (-1)^{n+1}a_n \\ 1 & 0 & \cdot & \cdot & (-1)^n a_{n-1} \\ 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & 0 & -a_2 \\ 0 & \cdot & \cdot & 0 & 1 & a_1 \end{pmatrix}, \quad (3.2)$$

(where a_i are the coefficients of $p(T)$). Thus, $(W, p(T))$ lies on the image of the Krichever map

$$\mathcal{Higgs}_X^\infty \hookrightarrow \mathcal{U}_X^\infty \times \mathbb{A}$$

if and only if the restriction of T to W takes values in $W \cdot \Omega$:

$$\begin{array}{ccc} \mathcal{O}_S((z))^n & \xrightarrow{T} & \mathcal{O}_S((z))^n \\ \uparrow & & \uparrow \\ W & \xrightarrow{T|_W} & W \cdot \Omega \end{array}$$

For rational points, this condition translates into $T(W) \subseteq W \cdot \Omega$.

Proof. The direct implication follows from the definition of the Krichever morphism. Let us see how to recover the geometric data from $(W, p(T)) \in \mathcal{U}_X^\infty(S) \times \mathbb{A}(S)$. By hypothesis, $T|_W$ takes values in $W \cdot \Omega$ and we already know that $W \in \mathcal{U}_X^\infty(S)$, so we can recover a rank n vector bundle on $X \times S$ together with its formal trivialization ϕ (see [1], [10]). By Proposition 3.3 we have that $W \cdot \Omega$ is the point in $\mathrm{Gr}(k((z))^n)(S)$ defined by $E \otimes_{\mathcal{O}_X} \omega_{X \times S/S}$; that is:

$$W \cdot \Omega = \varinjlim_m \pi_{S,*}(E \otimes_{\mathcal{O}_X} \omega_{X \times S/S})(m(x \times S)).$$

Therefore, $T|_W$ yields a morphism:

$$T|_W: \varinjlim_m \pi_{S,*} E(m(x \times S)) \rightarrow \varinjlim_m \pi_{S,*}(E \otimes_{\mathcal{O}_X} \omega_{X \times S/S})(m(x \times S)),$$

where $\pi_S: X \times S \rightarrow S$. Bearing in mind that T is $\mathcal{O}_S((z))$ -linear and the classical inverse Krichever correspondence, we can recover the Higgs field:

$$E \rightarrow E \otimes_{\mathcal{O}_X} \omega_{X \times S/S}.$$

The compatibility condition between φ and ϕ is granted from the construction. $\square$

This result allows us to prove the following theorem:

Theorem 3.4. *The functor $\mathcal{Higgs}_X^\infty$ is representable by a closed subscheme of $\mathcal{U}_X^\infty \times \mathbb{A}$.*

Proof. By Theorem 3.3, the image of:

$$\mathcal{Higgs}_X^\infty \hookrightarrow \mathcal{U}_X^\infty \times \mathbb{A}$$

consists of pairs $(W, p(T))$ such that $T|_W(W) \subseteq W \cdot \Omega$. By Proposition 3.3 we have that $W \cdot \Omega \in \text{Gr}(V)$. The statement is therefore a consequence of [1], which reads as follows.

Let S be a k -scheme and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$ -modules. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two quasi-coherent subsheaves of $\mathcal{F}$ such that locally $\mathcal{F}/\mathcal{F}_2 \simeq \varprojlim_i L_i$, where L_i are free coherent sheaves. Thus, the points $f: S' \rightarrow S$ such that $f^*(\mathcal{F}_1) \subseteq f^*(\mathcal{F}_2)$, as subsheaves of $f^*(\mathcal{F})$, are the points of a closed subset of S (see also the proof of [13]). $\square$

4. Spectral Construction.

Definition 4.1. We call $\text{Spf } k[[z]]$ a formal base curve. Given $p(T) \in \mathbb{A}(k)$, we define the *formal spectral curve* by:

$$\text{Spf } k[[z]][T]/p(T).$$

Let (E, ϕ, φ) be point in $\mathcal{Higgs}_X^\infty$ with values in S . Recall that the formal Hitchin map (see Definition 3.2):

$$\mathcal{H}^\infty: \mathcal{Higgs}_X^\infty \rightarrow \mathbb{A}$$

sends (E, ϕ, φ) to the characteristic polynomial $p_{\widehat{\varphi}}(T) \in \mathbb{A}(S)$ of the formal Higgs field $\widehat{\varphi}$.

Proposition 4.1. *Let S be a k -scheme. Given $(E, \phi, \varphi) \in \mathcal{Higgs}_X^\infty(S)$, ϕ induces a formal trivialization of E as a line bundle over the spectral curve X_φ :*

$$\phi_\varphi: \widehat{E}_{x \times S} \simeq \mathcal{O}_S[[z]][T]/p_{\widehat{\varphi}}(T),$$

such that the matrix of Definition 3.1 is the multiplication by T in $\mathcal{O}_S[[z]][T]/p_{\widehat{\varphi}}(T)$ w.r.t the basis $\{1, T, \dots, T^{n-1}\}$.

Proof. The proof uses the classical correspondence between a Higgs pair and the Jacobian of the spectral curve of [3]. The formal Higgs field $\widehat{\varphi}$ endows $\widehat{E}_{x \times S}$ with a line bundle structure over the formal spectral curve $\text{Spf } \mathcal{O}_S[[z]][T]/p_{\widehat{\varphi}}(T)$. We can therefore choose a trivialization:

$$\widehat{E}_{x \times S} \simeq \mathcal{O}_S[[z]][T]/p_{\widehat{\varphi}}(T).$$

12 *D. Hernández-Serrano, J. M. Muñoz Porras, F. J. Plaza Martín*

Using the given formal trivialization:

$$\phi: \widehat{E}_{x \times S} \simeq \mathcal{O}_S[[z]]^n$$

we can define an $\mathcal{O}_S[[z]]$ -algebra structure on $\mathcal{O}_S[[z]]^n$ and, because of the compatibility condition between ϕ and φ of Definition 3.1, we have that M corresponds to the multiplication by T in $\mathcal{O}_S[[z]][T]/p_{\widehat{\varphi}}(T)$. $\square$

4.1. *Relative Infinite Grassmannian.*

Observe that for each $p(T) \in \mathbb{A}(S)$ the $\mathcal{O}_S((z))$ -module $V_S = \mathcal{O}_S((z))^n$ can be enriched with a natural $\mathcal{O}_S((z))$ -algebra structure:

$$V_S \simeq \mathcal{O}_S((z))[T]/p(T).$$

Now let us take a rational point (E, ϕ, φ) in $\mathcal{Higgs}_X^\infty$ and let us denote by $\pi: X_\varphi \rightarrow X$ the spectral cover.

Using the formal trivialization of Proposition 4.1:

$$\phi_\varphi: \widehat{E}_x \simeq k[[z]][T]/p_{\widehat{\varphi}}(T)$$

it is easy to check that:

$$W = \phi_\varphi H^0(X_\varphi - \pi^{-1}(x), E)$$

is a rational point of the infinite Grassmannian $\mathrm{Gr}(k((z))[T]/p_{\widehat{\varphi}}(T))$.

The aim of this section is to allow $p(T)$ to vary, and this will lead us to define the relative infinite Grassmannian. The construction is similar to that of the standard infinite Grassmannian ([2]), so we shall omit the proofs (see [14] for further details).

Let $p^{univ}(T)$ denote the universal polynomial of $\mathbb{A}$; that is, the point corresponding to the identity morphism $\mathrm{Id}: \mathbb{A} \rightarrow \mathbb{A}$ and let us think of $\mathcal{O}_{\mathbb{A}}[[z]][T]/p^{univ}(T)$ as the ring of the universal formal spectral cover.

For this section let us denote:

$$V := \mathcal{O}_{\mathbb{A}}((z))[T]/p^{univ}(T) \quad V^+ := \mathcal{O}_{\mathbb{A}}[[z]][T]/p^{univ}(T)$$

as sheaves of $\mathcal{O}_{\mathbb{A}}$ -modules.

There exists a notion of commensurability for sub- $\mathcal{O}_{\mathbb{A}}$ -modules of V and it satisfies equivalent properties as those of the standard commensurability (see [2]). This notion endows V with a topology.

Definition 4.2. The relative infinite Grassmannian associated with the couple (V, V^+) is the functor that associates the following set with each $\mathbb{A}$ -scheme S :

$$\mathcal{G}r(V)(S) = \left\{ \begin{array}{l} \text{quasicoherent sub-}\mathcal{O}_S\text{-modules } W \subseteq V_S \text{ such that } V_S/W \text{ is} \\ \text{flat and for each point in } S \text{ there exists an open neighborhood } U \text{ and} \\ C \text{ commensurable with } V^+ \text{ such that } V_U/(W_U + C_U) = 0 \\ \text{and } W_U \cap C_U \text{ is loc. free of finite type} \end{array} \right\}$$

Theorem 4.1. $\mathcal{G}r(V)$ is representable by an $\mathbb{A}$ -scheme (thus, in particular by a k -scheme) that we shall still call $\mathcal{G}r(V)$.

Theorem 4.2. For all $\mathbb{A}$ -scheme S denote $V_S = V \widehat{\otimes}_{\mathcal{O}_{\mathbb{A}}} \mathcal{O}_S$. It holds that:

$$\mathcal{G}r(V_S) \simeq \mathcal{G}r(V) \times_{\mathbb{A}} S$$

In particular, for each rational point $p(T): \text{Spec } k \rightarrow \mathbb{A}$, we have:

$$\mathcal{G}r(V_k) = \mathcal{G}r(k((z))[T]/p(T)).$$

This means that:

$$\mathcal{G}r(\mathcal{O}_{\mathbb{A}}((z))[T]/p^{univ}(T)) \rightarrow \mathbb{A}$$

is a fibration of infinite Grassmannians.

Remark 4.1. Note that one has an isomorphism of $\mathcal{O}_{\mathbb{A}}$ -modules:

$$\mathcal{O}_{\mathbb{A}}((z))[T]/p^{univ}(T) \simeq \mathcal{O}_{\mathbb{A}}((z))^n.$$

Therefore, considering the trivial fibration of k -schemes:

$$\text{Gr}(k((z))^n) \times \mathbb{A} \rightarrow \mathbb{A}$$

one can obtain isomorphisms:

$$\text{Gr}(k((z))^n) \times \mathbb{A} \simeq \mathcal{G}r(\mathcal{O}_{\mathbb{A}}((z))^n) \simeq \mathcal{G}r(\mathcal{O}_{\mathbb{A}}((z))[T]/p^{univ}(T)). \quad (4.1)$$

4.2. Krichever map via the spectral construction.

Given a point $(E, \phi, \varphi) \in \mathcal{Higgs}_X^{\infty}(S)$, and using the formal trivialization ϕ_{φ} of Proposition 4.1, one obtains a well-defined Krichever map:

$$\begin{aligned} \mathcal{Higgs}_X^{\infty}(S) &\rightarrow \mathcal{G}r(\mathcal{O}_{\mathbb{A}}((z))[T]/p^{univ}(T))(S) \\ (E, \phi, \varphi) &\mapsto \phi_{\varphi}(\varinjlim_m \pi'_* E_{\varphi}(m)), \end{aligned}$$

where $\pi': X_{\varphi} \rightarrow S$. For rational points, the map is given by:

$$\begin{aligned} \mathcal{Higgs}_X^{\infty}(k) &\rightarrow \mathcal{G}r(\mathcal{O}_{\mathbb{A}}((z))[T]/p^{univ}(T))(k) \\ (E, \phi, \varphi) &\mapsto \phi_{\varphi} H^0(X_{\varphi} - \pi^{-1}(x), E_{\varphi}), \end{aligned}$$

$\pi: X_{\varphi} \rightarrow X$ being the spectral cover.

Bearing in mind the isomorphism of equation (4.1), it follows that this Krichever map is in fact equivalent to the Krichever map of Definition 3.3, and by Theorem 3.2 it is injective. The image is characterized in a similar way to that of Theorem 3.3.

See also [4] for another interpretation of the spectral construction.

Remark 4.2. Recently a preprint by A. Hodge and M. Mulase ([11]) has appeared containing interesting results about the Hitchin integrable systems related to our paper. Those authors make a precise study of the Hitchin integrable system for the symplectic group and study its connection with mirror symmetry. Finally they also study the Sp_{2m} -Hitchin system in terms of the Sato Grassmannian. We hope that the methods developed here can be used in the problems stated in that preprint, and more generally in the study of certain particular cases of the Langlands duality (e.g. [6]).

5. Equations of the moduli space of Higgs Pairs.

Let $p(T) \in \mathbb{A}(k)$ be a rational point and let us denote:

$$V_p = k((z))[T]/p(T) \quad V_p^+ = k[[z]][T]/p(T).$$

From now on, let us assume that V_p is a separable $k((z))$ -algebra with the following decomposition:

$$V_p \simeq V_1 \times \cdots \times V_r,$$

where $V_i = k((z))[T_i]/T_i^{n_i} - zu_i(T_i)$, $n_1 + \cdots + n_r = n$ and $u_i(T_i)$ are invertible in $k[[T_i]]$. In addition, $k((T_i))$ can be endowed with a $k((z))$ -algebra structure by sending z to $T_i^{n_i}u_i(T_i)^{-1}$. Consequently one obtains an isomorphism:

$$k((T_i)) \simeq V_i = k((z))[T_i]/T_i^{n_i} - zu(T_i) \quad (5.1)$$

This assumption is motivated by the following geometric fact:

Proposition 5.1. *Let us consider a rational point (E, φ, ϕ) in $\mathcal{Higgs}_X^\infty$, and let $\pi: X_\varphi \rightarrow X$ be the associated spectral cover. Let us assume that $\pi^{-1}(x) = \sum_{i=1}^r n_i y_i$, where $\sum_{i=1}^r n_i = n$ and $y_i \neq y_j$ for all $i \neq j$. We then have:*

$$\widehat{\mathcal{O}}_{X_\varphi, \pi^{-1}(x)} \simeq (k[[z]][T_1]/T_1^{n_1} - zu_1(T_1)) \times \cdots \times (k[[z]][T_r]/T_r^{n_r} - zu_r(T_r)),$$

where $u_i(T_i) \in k[[z]][T_i]$.

Proof. Indeed, $\widehat{\mathcal{O}}_x$ is a local, regular and complete ring with maximal ideal $\mathfrak{m}_x = (z)$, and $\widehat{\mathcal{O}}_{X_\varphi, y}$ is a regular, complete, separable $\widehat{\mathcal{O}}_{X, x}$ -algebra of dimension one. Thus, $\widehat{\mathcal{O}}_{X_\varphi, y}$ is a Dedekind domain and we have a primary decomposition:

$$\mathfrak{m}_x \mathcal{O}_{X_\varphi, y} = \mathfrak{m}_1^{n_1} \cdots \mathfrak{m}_r^{n_r},$$

where $\mathfrak{m}_i$ are the maximal ideals of the points:

$$\{y_1, \dots, y_r\} = \mathrm{Spec} \mathcal{O}_{X_\varphi, y}.$$

Moreover, if we denote by $\widehat{\mathcal{O}}_{y_i}$ the completion of $(\mathcal{O}_{X_\varphi, y})_{y_i}$, in $\widehat{\mathcal{O}}_{y_i}$ we have $\mathfrak{m}_x \mathcal{O}_{y_i} = \mathfrak{m}_i^{n_i}$. Since $\widehat{\mathcal{O}}_{y_i}$ are local, regular and complete k -algebras of dimension one, Cohen's Theorem implies that $\widehat{\mathcal{O}}_{y_i} \simeq k[[T_i]]$ and hence, in $\widehat{\mathcal{O}}_{y_i}$, we have:

$$z = T_i^{n_i} v_i(T_i),$$

$v_i(T_i)$ being an invertible element in $\widehat{\mathcal{O}}_{y_i}$. In addition, we know that $k[[z]] \rightarrow k[[T_i]]$ is an integral morphism, and hence we have a relation:

$$T_i^{n_i} + b_1^{(i)} T_i^{n_i-1} + \cdots + b_{n_i}^{(i)} = 0,$$

where $b_j^{(i)} \in k[[z]]$. That is:

$$k[[T_i]] \simeq k[[z]][T_i]/T_i^{n_i} + b_1^{(i)} T_i^{n_i-1} + \cdots + b_{n_i}^{(i)}.$$

Moreover, if we restrict ourselves to $z = 0$ we should obtain a point counted n_i times, which means that $b_j^{(i)} = z \bar{b}_j^{(i)}$, where $\bar{b}_j^{(i)} \in k[[z]]$. Therefore:

$$T_i^{n_i} = -z(\bar{b}_1^{(i)} T_i^{n_i-1} + \cdots + \bar{b}_{n_i}^{(i)}).$$

If we denote $v_i(T_i)^{-1}$ by $u_i(T_i)$, we have that $u_i(T_i)$ is a polynomial in T_i with coefficients in $k[[z]]$, and:

$$k[[T_i]] \simeq k[[z]][T_i]/T_i^{n_i} - z u_i(T_i).$$

The separability of V_p and the fact that the characteristic is zero, imply that if $u_i = u_j$ then $n_i \neq n_j$. Now, we can conclude:

$$k[[z]][T]/p_{\widehat{\varphi}}(T) \simeq k[[z]][T_1]/T_1^{n_1} - z u_1(T_1) \times \cdots \times k[[z]][T_r]/T_r^{n_r} - z u_r(T_r). \quad \square$$

Remark 5.1. Note that, since k has characteristic zero, there exists $\bar{v}_i \in k[[T_i]]$ such that $v_i = \bar{v}_i^{n_i}$. Therefore, $\bar{T}_i = T_i \bar{v}_i \in k[[T_i]]$ is a formal parameter at y_i and $z = \bar{T}_i^{n_i}$. That is:

$$\widehat{\mathcal{O}}_{y_i} \simeq \widehat{\mathcal{O}}_x[\bar{T}_i]/\bar{T}_i^{n_i} - z.$$

This means that we require a change in parameter, and hence a change in polynomial in particular. Since we want to allow $p(T)$ to vary on $\mathbb{A}$, we are not allowed to make this change in parameter.

In order to give the equations describing $\mathcal{Higgs}_X^\infty$, we first need to adapt some results concerning Tau and Baker-Akhiezer functions given in [2, 13, 12].

5.1. Formal Jacobian and Abel morphism.

Because of the isomorphisms of equation (5.1) and following Theorem 4.10 of [2] and [12] (pag. 3), it can be proved that the functor $\underline{V}_p^*$ of invertible elements in V_p is representable by a formal group k -scheme whose connected component at the origin, Γ_{V_p} , decomposes as:

$$\Gamma_{V_p} = \Gamma_{V_p}^- \times \mathbb{G}_m^r \times \Gamma_{V_p}^+$$

and

$$\Gamma_{V_p}^\pm \simeq \Gamma_{V_1}^\pm \times \cdots \times \Gamma_{V_r}^\pm.$$

Moreover, the formal Jacobian of the formal spectral curve $\widehat{X}_V$, $\mathcal{J}(\widehat{X}_V)$, is isomorphic to $\Gamma_{V_p}^-$ (see Theorem 4.14 of [2] and also [12]).

Let $\widehat{\mathbb{A}}_\infty$ be the formal group scheme:

$$\widehat{\mathbb{A}}_\infty := \varinjlim_n \operatorname{Spec} k[[t_1, \dots, t_n]]$$

endowed with the additive group law. Let us denote by $k\{\{t_1, t_2, \dots\}\}$ the ring $\mathcal{O}_{\widehat{\mathbb{A}}_\infty} = \varprojlim_n k[[t_1, \dots, t_n]]$ and let us set:

$$\mathbb{A}_\infty^r := \widehat{\mathbb{A}}_\infty \times \dots \times \widehat{\mathbb{A}}_\infty.$$

The exponential map is the isomorphism of formal group k -schemes (see Definition 4.11 of [2]):

$$\begin{aligned} \exp: \widehat{\mathbb{A}}_\infty^r &\rightarrow \mathcal{J}(\widehat{X}_V) \\ (\{a_i^{(1)}\}_{i>0}, \dots, \{a_i^{(r)}\}_{i>0}) &\mapsto \left(\exp\left(\sum_{i>0} \frac{a_i^{(1)}}{T_1^i}\right), \dots, \exp\left(\sum_{i>0} \frac{a_i^{(r)}}{T_r^i}\right) \right). \end{aligned}$$

We can now identify $\mathcal{J}(\widehat{X}_V)$ with the formal spectrum of the ring:

$$k\{\{t_1^{(1)}, \dots\}\} \hat{\otimes} \dots \hat{\otimes} k\{\{t_1^{(r)}, \dots\}\},$$

and its universal element will be denoted by:

$$v = \prod_{i=1}^r \exp\left(\sum_{j \geq 1} \frac{t_j^{(i)}}{T_i^j}\right).$$

The Abel morphism of degree one (see [12]):

$$\phi_1: \widehat{X}_V \rightarrow \mathcal{J}(\widehat{X}_V)$$

is the morphism corresponding to the r -uple of series:

$$\left((1 - \frac{\bar{T}_1}{T_1})^{-1}, \dots, (1 - \frac{\bar{T}_r}{T_r})^{-1} \right),$$

where we distinguish the indeterminates:

$$\bar{T}_i \in \mathcal{O}_{\widehat{X}_V} = k[[\bar{T}_1]] \times \dots \times k[[\bar{T}_r]] \simeq V_p^+$$

and $T_i \in \mathcal{O}_{\mathcal{J}(\widehat{X}_V)}$. Equivalently, the Abel morphism is the morphism induced by the following ring homomorphism:

$$\begin{aligned} k\{\{t_1^{(1)}, t_2^{(1)}, \dots\}\} \hat{\otimes} \dots \hat{\otimes} k\{\{t_1^{(r)}, t_2^{(r)}, \dots\}\} &\rightarrow k[[\bar{T}_1]] \times \dots \times k[[\bar{T}_r]] \\ t_i^{(j)} &\mapsto \bar{T}_j^{(i)} \end{aligned}$$

5.2. Tau and Baker-Akhiezer functions.

Let us consider the fibration of infinite Grassmannians:

$$\mathrm{Gr}(k((z))^n) \times \mathbb{A} \rightarrow \mathbb{A}$$

and let $p(T)$ be a rational point of $\mathbb{A}$. The fiber at $p(T) \in \mathbb{A}(k)$ is $\mathrm{Gr}(V_p)(k)$ (see section 4.1), where $V_p = k((z))[T]/p(T)$.

Let us now consider the natural action of Γ_{V_p} in $\mathrm{Gr}(V_p)$:

$$\begin{aligned} \mu: \Gamma_{V_p} \times \mathrm{Gr}(V_p) &\rightarrow \mathrm{Gr}(V_p) \\ (g_\bullet, W) &\mapsto g_\bullet \cdot W, \end{aligned}$$

and let us define the Poincaré sheaf on $\Gamma_{V_p} \times \mathrm{Gr}(V_p)$ by:

$$\mathcal{P} := \mu^* \mathrm{Det}_{V_p}^*.$$

For each rational point $W \in \mathrm{Gr}(V_p)$, one has an invertible sheaf on Γ_{V_p} :

$$\tilde{\mathcal{L}}_\tau(W) := \mathcal{P}|_{\Gamma_{V_p} \times \{W\}}$$

and a natural homomorphism:

$$\mathrm{H}^0(\Gamma_{V_p} \times \mathrm{Gr}(V_p), \mathcal{P}) \rightarrow \mathrm{H}^0(\Gamma_{V_p} \times \{W\}, \tilde{\mathcal{L}}_\tau(W)). \quad (5.2)$$

Definition 5.1. We use the term τ -section of a point W , $\tilde{\tau}_W$, to refer to the image of $\mu^* \Omega_+$ under the morphism (5.2), where Ω_+ is the canonical global section of $\mathrm{Det}_{V_p}^*$ of Definition 3.14 in [2].

Note that $\tilde{\tau}_W$ is not a true function over $\Gamma_{V_p} \times \{W\}$, because $\tilde{\mathcal{L}}_\tau(W)$ is in general nontrivial. The algebraic analogue of the tau function defined by Sato-Sato ([15]) and Segal-Wilson ([16]) is obtained by restricting $\tilde{\mathcal{L}}_\tau(W)$ to the formal subgroup $\Gamma_{V_p}^- \simeq \mathcal{J}(\hat{X}_V) \subset \Gamma_{V_p}$.

Let us define $\mathcal{L}_\tau(W) := \tilde{\mathcal{L}}_\tau(W)|_{\Gamma_{V_p}^- \times \{W\}}$, which is trivial over $\Gamma_{V_p}^- \times \{W\}$. Indeed a trivialization of $\mathcal{L}_\tau(W)$ can be given by the global section:

$$\sigma_0(g_\bullet) := g_\bullet \cdot \rho_W,$$

where ρ_W is a non-zero element in the fiber of $\mathcal{L}_\tau(W)$ over the point $(1, W) \in \Gamma_{V_p} \times W$. The τ -function is defined as the trivialization of the restriction of $\tilde{\tau}_W$ to Γ_{V_p} .

Definition 5.2. The τ -function of the point W , τ_W , is the function:

$$\tau_W \in \mathcal{O}_{\Gamma_{V_p}^-} \simeq \mathcal{O}_{\mathcal{J}(\hat{X}_V)} \simeq k\{\{t_1^{(1)}, \dots\}\} \hat{\otimes} \dots \hat{\otimes} k\{\{t_1^{(r)}, \dots\}\}$$

defined by:

$$\tau_W(g_\bullet) = \frac{\tilde{\tau}_W(g_\bullet)}{\sigma_0(g_\bullet)} = \frac{\mu^* \Omega_+(g_\bullet)}{\sigma_0(g_\bullet)} = \frac{\Omega_+(g_\bullet \cdot W)}{g_\bullet \cdot \rho_W}$$

18 *D. Hernández-Serrano, J. M. Muñoz Porras, F. J. Plaza Martín*

for $g_\bullet = \prod_{i=1}^r \exp(\sum_{j \geq 1} \frac{t_j^{(i)}}{T_i^j}) \in \mathcal{J}(\widehat{X}_V) \simeq \Gamma_{V_p}^-$.

Remark 5.2. In particular, over the formal base curve $\widehat{X} = \text{Spf } k[[z]]$ the τ function of a point $\Omega \in \text{Gr}(k((z)))$ is the function in $\mathcal{J}(\widehat{X}) \simeq \Gamma^- \simeq \text{Spf } k\{\{t_1, \dots\}\}$ defined by ([2]):

$$\tau_\Omega(g) = \frac{\tilde{\tau}_\Omega(g)}{\sigma_0(g)} = \frac{\mu^* \Omega_+(g)}{\sigma_0(g)} = \frac{\Omega_+(gW)}{g\rho_W},$$

where $g = \exp(\sum_{j \geq 1} \frac{t_j}{z^j}) \in \mathcal{J}(\widehat{X}) \simeq \Gamma^-$.

Let us now consider the composition:

$$\tilde{\beta}: \widehat{X}_V \times \Gamma_{V_p} \times \text{Gr}(V_p) \xrightarrow{\phi_1 \times \text{Id}} \Gamma_{V_p} \times \Gamma_{V_p} \times \text{Gr}(V_p) \xrightarrow{m \times \text{Id}} \Gamma_{V_p} \times \text{Gr}(V_p)$$

where $\phi_1: \widehat{X}_V \rightarrow \Gamma_{V_p}$ is the Abel morphism of degree one (with values in $\Gamma_{V_p}^- \simeq \mathcal{J}(\widehat{X}_V) \subset \Gamma_{V_p}$) and m is the group law in Γ_{V_p} . Given a point $W \in \text{Gr}(V_p)$, let us denote by $\tilde{\beta}_W$ the restriction of $\tilde{\beta}$ to $\widehat{X}_V \times \Gamma_{V_p} \times \{W\}$.

Definition 5.3. The BA-section of $W \in \text{Gr}(V_p)$ is $\tilde{\psi}_W := v^{-1} \cdot \beta_W^*(\tilde{\tau}_W)$, where:

$$\beta_W^*: H^0(\Gamma_{V_p} \times \{W\}, \tilde{\mathcal{L}}_\tau(W)) \rightarrow H^0(\widehat{X}_V \times \Gamma_{V_p} \times \{W\}, \tilde{\beta}_W^* \tilde{\mathcal{L}}_\tau(W))$$

is the morphism induced by $\tilde{\beta}_W^*$. That is, let us choose it in such a way that:

$$\widetilde{\psi_W}|_{x_0^{(i)} \times \Gamma_{V_p}} = v_i^{-1},$$

where $x_0^{(i)}$ is the origin of $\widehat{X}_{V_i}$ and v^{-1} is v_i^{-1} in the component $\widehat{X}_{V_i}$.

The BA-function of a point $W \in \text{Gr}(V_p)$ is defined by the following formula:

$$\psi_W(T_\bullet, g_\bullet) = v^{-1} \frac{\tau_W(g_\bullet \cdot \phi_1(T_\bullet))}{\tau_W(g_\bullet)},$$

where $T_\bullet = (T_1, \dots, T_r)$ and $g_\bullet \in \Gamma_{V_p}^-$.

In order to give an explicit expression of this function, we identify the formal Jacobian $\mathcal{J}(\widehat{X}_V)$ with $\widehat{\mathbb{A}}_\infty^r$ (via the exponential isomorphism) and we use the Abel morphism:

$$\widehat{X}_V \xrightarrow{\phi_1} \mathcal{J}(\widehat{X}_V),$$

which sends T_j to the point in $\mathcal{J}(\widehat{X}_V)$ with coordinates:

$$[T_j] := ((0, \dots), \dots, (T_j, \frac{T_j^2}{2}, \frac{T_j^3}{3}, \dots), \dots, (0, \dots)),$$

or, equivalently, this map is induced by the ring homomorphism:

$$k\{\{t_1^{(1)}, \dots\}\} \hat{\otimes} \dots \hat{\otimes} k\{\{t_1^{(r)}, \dots\}\} \rightarrow k[[T_1]] \times \dots \times k[[T_r]]$$

$$t_i^{(j)} \mapsto (0, \dots, 0, \frac{T_j^{(i)}}{i}, 0, \dots, 0).$$

It is clear that we have an addition map:

$$\begin{aligned}\widehat{X}_V \times \mathcal{J}(\widehat{X}_V) &\rightarrow \mathcal{J}(\widehat{X}_V) \\ (T_\bullet, t_\bullet) &\mapsto t_\bullet + [T_\bullet]\end{aligned}$$

where $T_\bullet = (T_1, \dots, T_r)$, $t_\bullet = (t^{(1)}, \dots, t^{(r)})$ and $t_\bullet + [T_\bullet]$ denotes the point in $\mathcal{J}(\widehat{X}_V)$ with coordinates $(\dots, t_i^{(j)} + \frac{T_i^j}{i}, \dots)$.

Remark 5.3. If $\Omega \in \text{Gr}(k((z)))$, its BA-function is $([2, 13])$:

$$\psi_\Omega(z, t) = \exp\left(-\sum_{j \geq 1} \frac{t_j}{z^j}\right) \cdot \left(\frac{\tau_\Omega(t + [z])}{\tau_\Omega(t)}\right).$$

Remark 5.4. In the case of points in the Grassmannian of $k((z))$, the Tau-function of a point $\Omega \in \text{Gr}(k((z)))$ generates the subspace $\Omega([13])$. Nevertheless, this is no longer true for a point $W \in \text{Gr}(V_p)$. In order to solve this problem, the following definition is used in [12] and [9].

Let us recall the decomposition $V_p \simeq V_1 \times \dots \times V_r$.

Definition 5.4. The u -th BA-function of a point $W \in \text{Gr}(V_p)$ is the function:

$$\psi_{u,W}(T_\bullet, t_\bullet) := (\xi_{u1} \exp\left(-\sum_{j \geq 1} \frac{t_j^{(1)}}{T_1^j}\right) \frac{\tau_{W_{u1}}(t_\bullet + [T_1])}{\tau_W(t_\bullet)}, \dots, \xi_{ur} \exp\left(-\sum_{j \geq 1} \frac{t_j^{(r)}}{T_r^j}\right) \frac{\tau_{W_{ur}}(t_\bullet + [T_r])}{\tau_W(t_\bullet)}),$$

where:

- $1 \leq u \leq r$.
- $W_{uv} := (1, \dots, T_u, \dots, T_v^{-1}, \dots, 1) \cdot W$.
- $t_\bullet + [T_v] := (t^{(1)}, \dots, t^{(v)} + [T_v], \dots, t^{(n)})$.
- ξ_{ui} is -1 if $i > u$ and 1 if $i \leq u$.

Now, following [12] one can see that W can be generated (as an infinite subspace of V_p) by these u -th BA-functions.

Theorem 5.1. ([12], Theorem 3.6). Let us take $W \in \text{Gr}^0(V_p)$. We thus have:

$$\psi_{u,W}(T_\bullet, t_\bullet) = (1, \dots, T_u, \dots, 1) \cdot \sum_{i \geq 0} (\psi_{u,W}^{i,1}(T_1), \dots, \psi_{u,W}^{i,r}(T_r)) p_{ui,W}(t_\bullet),$$

where:

$$\{(\psi_{u,W}^{i,1}(T_1), \dots, \psi_{u,W}^{i,r}(T_r)) \mid i \geq 0, 1 \leq u \leq r\}$$

is a basis for W and $p_{ui,W}(t_\bullet)$ are functions in $t_\bullet$.

Remark 5.5. Let us recall that the superindex 0 of $\text{Gr}^0(V_p)$ tells us that we are taking the connected component of the Grassmannian on which the index function of W takes the value 0 (see Definition 3.3 of [2]).

Remark 5.6. To study the m -connected component ($m > 0$), $\mathrm{Gr}^m(V_p)$, we need to choose an element v_m such that $\dim_k V_p^+ / v_m V_p^+ = m$ in order to obtain an isomorphism $\mathrm{Gr}^m(V_p) \simeq \mathrm{Gr}^0(V_p)$ that will allow us to define the global section Ω_+^m of the dual of the determinant bundle of $\mathrm{Gr}^m(V_p)$ as the image of the canonical section Ω_+ of $\mathrm{Det}_{V_p}^*$ (see [2] and [13]). Let us set v_m as follows:

- for $m \leq \frac{1}{2}(r-n)$, let p, q, s, t be integer numbers defined by $-m = q \cdot (n-r) + p$, $0 \leq p < n-r$, $p = s \cdot r + t$, $0 \leq t < r$. Therefore, we set:

$$v_m := (z^{-1} \cdot T_\bullet)^q T_1^{s+1} \dots T_t^{s+1} T_{t+1}^s \dots T_r^s.$$

- for $m > \frac{1}{2}(r-n)$, we set:

$$v_m := (z^{-1} \cdot T_\bullet) \cdot v_{r-n-m}^{-1}.$$

Theorem 5.2. ([12], Theorem 3.7). *If $W \in \mathrm{Gr}^m(V_p)$ we have:*

$$\psi_{u,W}(T_\bullet, t_\bullet) = v_m^{-1}(1, \dots, T_u, \dots, 1) \cdot \sum_{i \geq 0} (\psi_{u,W}^{(i,1)}(T_1), \dots, \psi_{u,W}^{(i,r)}(T_r)) p_{ui,W}(t_\bullet),$$

where

$$\{(\psi_{u,W}^{(i,1)}(T_1), \dots, \psi_{u,W}^{(i,r)}(T_r)) \mid i > 0, 1 \leq u \leq t\}$$

is a basis for W and $p_{ui,W}(t_\bullet)$ are functions in $t_\bullet$.

In particular, an element of V_p lies in W if and only if it can be expressed as a linear combination of:

$$\psi_{1,W}(T_\bullet, t_\bullet), \dots, \psi_{r,W}(T_\bullet, t_\bullet)$$

for certain values of the parameters $t_\bullet$.

Remark 5.7. If $\Omega \in \mathrm{Gr}^s(k((z)))$, then ([13]):

$$\psi_\Omega(z, t) = z^{1-s} \sum_{i \geq 0} \psi_\Omega^{(i)}(z) p_i(t),$$

where $\{\psi_\Omega^{(i)}(z)\}_i$ is a basis for Ω and $p_i(t)$ are functions in t . In particular, if Ω is the point defined by the canonical line bundle ω_X , then $s = g - 1$, where g is the genus of X . If we denote Ω^{-1} as the point defined by ω_X^{-1} , then $s = 1 - 3g$.

5.2.1. Adjoint Baker function.

Since V_p is a finite separable $k((z))$ -algebra, it carries the metric of the trace:

$$\mathrm{Tr}: V_p \times V_p \rightarrow k((z)),$$

which is non-degenerate. Therefore, V_p can be endowed with the non-degenerate pairing:

$$\begin{aligned} \mathrm{T}_2: V_p \times V_p &\rightarrow k \\ (a, b) &\mapsto \mathrm{Res}_{z=0}(\mathrm{Tr}(a, b))dz. \end{aligned}$$

The pairing T_2 induces an isomorphism of k -schemes:

$$\begin{aligned} R: \operatorname{Gr}(V_p) &\rightarrow \operatorname{Gr}(V_p) \\ W &\mapsto W^\perp, \end{aligned}$$

where $W^\perp$ is written for the orthogonal of W w.r.t. T_2 (see [12]) and we have the following properties:

- $R(\operatorname{Gr}^m(V_p)) = \operatorname{Gr}^{-m}(V_p)$.
- $R^* \operatorname{Det}_{V_p} \simeq \operatorname{Det}_{V_p}$.
- $(g \cdot W)^\perp = g^{-1} \cdot W^\perp$, for $W \in \operatorname{Gr}(V_p)$ and $g \in \mathcal{J}(\widehat{X}_V)$.
- $R^* \Omega_+^m = \Omega_+^{-m}$.
- $\tau_{W^\perp}(g) = \tau_W(g^{-1})$.
- $(W_{uv})^\perp = (W^\perp)_{vu}$.

Definition 5.5. The u -th Baker-Akhiezer function of a point $W \in \operatorname{Gr}(V_p)$ is:

$$\psi_{u,W}^*(T_\bullet, t_\bullet) := \psi_{u,W^\perp}(T_\bullet, -t_\bullet)$$

$$\begin{aligned} \psi_{u,W}^*(T_\bullet, t_\bullet) &:= \\ &= (\xi_{u1} \exp(\sum_{j \geq 1} \frac{t_j^{(1)}}{T_1^j}) \frac{\tau_{W_{u1}^\perp}(-t_\bullet + [T_1])}{\tau_W^\perp(-t_\bullet)}, \dots, \xi_{un} \exp(\sum_{j \geq 1} \frac{t_j^{(n)}}{T_n^j}) \frac{\tau_{W_{un}^\perp}(-t_\bullet + [T_r])}{\tau_W^\perp(-t_\bullet)}) = \\ &= (\xi_{u1} \exp(\sum_{j \geq 1} \frac{t_j^{(1)}}{T_1^j}) \frac{\tau_{W_{1u}}(t_\bullet - [T_1])}{\tau_W(t_\bullet)}, \dots, \xi_{ur} \exp(\sum_{j \geq 1} \frac{t_j^{(r)}}{T_r^j}) \frac{\tau_{W_{ru}}(t_\bullet - [T_r])}{\tau_W(t_\bullet)}). \end{aligned}$$

Theorem 5.3. Let us take $W \in \operatorname{Gr}^m(V_p)$. We have:

$$\psi_{u,W}^*(T_\bullet, t_\bullet) = v_{r-n-m}^{-1}(1, \dots, T_u, \dots, 1) \cdot \sum_{i > 0} (\psi_{u,W}^{*(i,1)}(T_1), \dots, \psi_{u,W}^{*(i,r)}(T_r)) p_{ui,W}^*(t_\bullet),$$

where:

$$\{(\psi_{u,W}^{*(i,1)}(T_1), \dots, \psi_{u,W}^{*(i,r)}(T_r)) \mid i > 0, 1 \leq u \leq r\}$$

is a basis for $W^\perp$ and $p_{ui,W}^*(t_\bullet)$ are functions in $t_\bullet$.

In particular, an element of V_p lies in $W^\perp$ if and only if it can be expressed as a linear combination of:

$$\psi_{1,W}^*(T_\bullet, t_\bullet), \dots, \psi_{r,W}^*(T_\bullet, t_\bullet)$$

for certain values of the parameters $t_\bullet$.

5.3. Product of BA-functions.

Let us take $\Omega \in \text{Gr}(k((z)))$ and $W \in \text{Gr}(V_p)$. Let us consider its BA-functions:

$$\psi_\Omega(z, t) \quad \psi_W(T_\bullet, t_\bullet)$$

(which are functions of $\widehat{X} \times \Gamma^-$ and $\widehat{X}_V \times \Gamma_{V_p}^-$ respectively).

The graph, $\widehat{X}_V \rightarrow \widehat{X}_V \times \widehat{X}$, of the morphism $\widehat{\pi}: \widehat{X}_V \rightarrow \widehat{X}$ induces:

$$f: \widehat{X}_V \times \Gamma_{V_p}^- \times \Gamma^- \rightarrow (\widehat{X}_V \times \Gamma_{V_p}^-) \times (\widehat{X} \times \Gamma^-).$$

Remark 5.8. Note that the graph is the morphism induced by the ring homomorphism $\mathcal{O}_{\widehat{X}} \times \mathcal{O}_{\widehat{X}_V} \rightarrow \mathcal{O}_{\widehat{X}_V}$:

$$\begin{aligned} k[[z]] \otimes_k (k[[T_1]] \times \cdots \times k[[T_r]]) &\rightarrow k[[T_1]] \times \cdots \times k[[T_r]] \\ z \otimes_k (T_1, \dots, T_r) &\mapsto (zT_1, \dots, zT_r), \end{aligned}$$

and this expression makes sense because we know how to express z in terms of T_i by equation (5.1).

Definition 5.6. The product of the BA-functions of $\Omega \in \text{Gr}(k((z)))$ and $W \in \text{Gr}(V_p)$ is defined by:

$$\psi_W(T_\bullet, t_\bullet) * \psi_\Omega(z, t) := f^*(\psi_W(T_\bullet, t_\bullet) \otimes_k \psi_\Omega(z, t))$$

The product of $\psi_\Omega(z, t)$ by the u -th BA-function of W , $\psi_{u,W}(T_\bullet, t_\bullet)$, can be defined in similar way.

Proposition 5.2. Let $W \in \text{Gr}(V_p)$ and let $\Omega \in \text{Gr}(k((z)))$ be such that $W \cdot \Omega \in \text{Gr}(V_p)$. We thus have that:

$$\{\psi_\Omega(z, t) * \psi_{1,W}(T_\bullet, t_\bullet), \dots, \psi_\Omega(z, t) * \psi_{r,W}(T_\bullet, t_\bullet)\}$$

is a generating system for $\Omega \cdot W$. Explicitly:

$$\psi_\Omega(z, t) * \psi_{u,W}(T_\bullet, t_\bullet) = (\psi_\Omega(z, t) \cdot \psi_{u,W}^{(1)}(T_1, t_\bullet), \dots, \psi_\Omega(z, t) \cdot \psi_{u,W}^{(r)}(T_r, t_\bullet)),$$

where

$$\psi_\Omega(z, t) = \exp\left(-\sum_{j \geq 1} \frac{t_j}{z^j}\right) \cdot \left(\frac{\tau_\Omega(t + [z])}{\tau_\Omega(t)}\right),$$

and

$$\psi_{u,W}^{(i)} = \xi_{ui} \exp\left(-\sum_{j \geq 1} \frac{t_j^{(i)}}{T_i^j}\right) \frac{\tau_{W_{ui}}(t_\bullet + [T_i])}{\tau_W(t_\bullet)}.$$

Proof. These functions are a generating system by Theorem 5.2 and Remark 5.7. The explicit formula follows from Remark 5.3, Definition 5.4, and the definition of the graph morphism. $\square$

5.4. Equations for $\mathcal{Higgs}_X^\infty$.

Theorem 3.3 says that a rational point $(W, p(T))$ of $\mathcal{U}_X^\infty \times \mathbb{A}$ lies on the image of:

$$\mathcal{Higgs}_X^\infty \hookrightarrow \mathcal{U}_X^\infty \times \mathbb{A}$$

if and only if $T(W) \subseteq W \cdot \Omega$, where $\Omega \in \text{Gr}(k((z)))$ is the point defined by the canonical line bundle ω_X , and T is the multiplication by T in V_p . Bearing in mind the decomposition $V_p \simeq V_1 \times \cdots \times V_r$, the image of T is $T_\bullet = (T_1, \dots, T_r)$. Therefore, if we think of T as an operator in V_p it is easy to see that T is self-adjoint w.r.t. the pairing T_2 defined in section 5.2.1; that is:

$$T_2(T \cdot v, v') = T_2(v, T \cdot v') \quad \forall v, v' \in V_p.$$

Theorem 5.4. *Let $(W, p(T)) \in \mathcal{U}_X^\infty \times \mathbb{A}$ be a rational point, $\Omega \in \text{Gr}^{g-1}(k((z)))$, and let us assume that $W \in \text{Gr}^m(V_p)$ (with $m \neq \frac{1}{2}(r-n)$).*

$(W, p(T))$ lies in the image of $\mathcal{Higgs}_X^\infty \hookrightarrow \mathcal{U}_X^\infty \times \mathbb{A}$ if and only if:

$$T_2\left(\frac{v_{r-n-m} \cdot T_\bullet \cdot \psi_{u,W}^*(T_\bullet, t_\bullet)}{(1, \dots, T_u, \dots, 1)}, \frac{v_m \cdot \psi_{\Omega^{-1}}(z, t) \cdot \psi_{v,W}(T_\bullet, t_\bullet)}{z^{3g} \cdot (1, \dots, T_v, \dots, 1)}\right) = 0$$

for all $1 \leq u, v \leq r$.

Proof. By theorem 3.3, $(W, p(T)) \in \mathcal{U}_X^\infty \times \mathbb{A}$ lies on the image of $\mathcal{Higgs}_X^\infty \hookrightarrow \mathcal{U}_X^\infty \times \mathbb{A}$ if and only if $T \cdot W \subseteq W \cdot \Omega$. We now have:

$$T \cdot W \subseteq W \cdot \Omega \iff W \cdot \Omega^{-1} \subseteq T^{-1} \cdot W \iff (T^{-1} \cdot W)^\perp \subseteq (W \cdot \Omega^{-1})^\perp,$$

where $\perp$ denotes the orthogonal w.r.t. T_2 .

Because T is self-adjoint, we have:

$$\begin{aligned} (T^{-1} \cdot W)^\perp &:= \{v \in V_p \mid T_2(v, T^{-1} \cdot w) = 0 \quad \forall w \in W\} = \\ &= \{v \in V_p \mid T_2(T^{-1} \cdot v, w) = 0 \quad \forall w \in W\} = \\ &= \{v \in V_p \mid T^{-1} \cdot v \in W^\perp\} = T \cdot W^\perp. \end{aligned}$$

Since multiplying by T corresponds to the multiplication by $T_\bullet$, using Theorem 5.3 we have that:

$$\psi_{1,W}^*(T_\bullet, t_\bullet), \dots, \psi_{r,W}^*(T_\bullet, t_\bullet)$$

is a generating system for $W^\perp$. Therefore:

$$T_\bullet \cdot \psi_{1,W}^*(T_\bullet, t_\bullet), \dots, T_\bullet \cdot \psi_{r,W}^*(T_\bullet, t_\bullet)$$

is a generating system for $T \cdot W^\perp$ and (by Theorem 5.3):

$$\begin{aligned} T_\bullet \cdot \psi_{u,W}^*(T_\bullet, t_\bullet) &= \\ &= v_{r-n-m}^{-1}(1, \dots, T_u, \dots, 1) \cdot \sum_{i>0} (T_1 \cdot \psi_{u,W}^{*(i,1)}(T_1), \dots, T_r \cdot \psi_{u,W}^{*(i,r)}(T_r)) p_{ui,W}^*(t_\bullet). \end{aligned}$$

Since T_2 is non-degenerate and because of Remark 5.7 and Theorems 5.2 and 5.3, we conclude that $T \cdot W^\perp \subseteq (\Omega^{-1} \cdot W)^\perp$ if and only if the following set of equations is satisfied:

$$T_2\left(\frac{v_{r-n-m} \cdot T_\bullet \cdot \psi_{u,W}^*(T_\bullet, t_\bullet)}{(1, \dots, T_u, \dots, 1)}, \frac{v_m \cdot \psi_{\Omega^{-1}}(z, t) \cdot \psi_{v,W}(T_\bullet, t_\bullet)}{z^{3g} \cdot (1, \dots, T_v, \dots, 1)}\right) = 0$$

for all $1 \leq u, v \leq r$. $\square$

5.5. Equations for irreducible polynomials.

In this section, we compute the equations describing $\mathcal{Higgs}_X^\infty$ for the case in which the spectral cover $\pi: X_\varphi \rightarrow X$ is totally ramified at $x \in X$ (that is, $\pi^{-1}(x) = ny$). These equations will be given in terms of the characteristic coefficients of the formal Higgs field.

In our formal setting, to assume this hypothesis implies that the characteristic polynomial of the formal Higgs field is irreducible. Henceforth, let us assume that we are given an irreducible polynomial:

$$p(T) = T^n - a_1 T^{n-1} + \dots + (-1)^n a_n.$$

The multiplication by T in $V_p = k((z))[T]/p(T)$ w.r.t. the basis $\{1, T, \dots, T^{n-1}\}$ corresponds to the matrix:

$$T = \begin{pmatrix} 0 & \cdot & \cdot & 0 & (-1)^{n+1} a_n \\ 1 & 0 & \cdot & \cdot & (-1)^n a_{n-1} \\ 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & 0 & -a_2 \\ 0 & \cdot & \cdot & 0 & 1 & a_1 \end{pmatrix}.$$

By equation (5.1) we are in the following case:

$$V_p = k((z))[T]/p(T) \simeq k((z))[T]/T^n - zu(T),$$

such that $u(T)$ can be computed explicitly in terms of $a_1, \dots, a_n$. Note in addition that multiplying by T^{-1} makes sense in V_p (but not in V_p^+).

If $W \in \text{Gr}^m(V_p)$, then its BA-function is given by:

$$\psi_W(z, T, t_n) = T^{1-m} \sum_{i>0} \psi_W^{(i)}(z, T) p_i(t_n), \quad (5.3)$$

where $\{\psi_W^{(i)}(z, T)\}_i$ is a basis for W and $p_i(t_n)$ are functions in t_n (see Remark 5.7 and note that we are writing t_n for the coordinates of Γ_{V_p}). Observe that the orthogonal $W^\perp$ w.r.t. the metric T_2 defined in section 5.2.1 lies on $\text{Gr}^{-m}(V_p)$. We therefore have:

$$\psi_W^*(T, t_n) = T^{1+m} \sum_{i>0} \psi_W^{*,(i)}(T) p_i^*(t_n), \quad (5.4)$$

where $\{\psi_W^{*,(i)}(T)\}_i$ is a basis for $W^\perp$ and $p_i^*(t_n)$ are functions in t_n .

Additionally, as in section 5.3 we can define the product of the BA-function $\Psi_{\Omega^{-1}}(z, t)$ of $\Omega^{-1} \in \text{Gr}(k((z)))$ by the BA-function $\Psi_W(z, T, t_n)$ of $W \in \text{Gr}(V_p)$.

In order to compute explicit equations for $\mathcal{Higgs}_X^\infty$, BA-functions will be thought of as mere functions on V_p ; that is, as polynomials on T with coefficients in $k((z))$. In this way, we set:

$$\Psi_W(z, T) = \sum_{i=0}^{n-1} \Psi_{W,i}(z) T^i, \quad \Psi_{W,i}(z) \in k((z)) \quad (5.5)$$

$$\Psi_W^*(z, T) = \sum_{i=0}^{n-1} \Psi_{W,i}^*(z) T^i, \quad \Psi_{W,i}^*(z) \in k((z)). \quad (5.6)$$

Let (E, φ, ϕ) be a rational point of $\mathcal{Higgs}_X^\infty$ and let:

$$p_\varphi(T) = T^n - a_1 T^{n-1} + \cdots + (-1)^n a_n \in \mathbb{A}(k)$$

be its formal characteristic polynomial. Newton-Girard identities (see Section I.2 of [8]) express $\text{Tr}(T^j)$ in terms of $a_i = \text{Tr}(\Lambda^i T)$ as follows:

$$\text{Tr}(T^j) = \begin{vmatrix} a_1 & 1 & 0 & \cdot & \cdot & \cdot \\ 2a_2 & a_1 & 1 & 0 & \cdot & \cdot \\ 3a_3 & a_2 & a_1 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_1 & 1 \\ ka_j & a_{j-1} & a_{j-2} & \cdot & \cdot & a_1 \end{vmatrix} \in k((z)), \quad \forall j \geq 0 \quad (5.7)$$

where $a_j = 0$ for $j > n$.

Let us now consider the subscheme $\mathcal{Higgs}_X^\infty(n)$ of $\mathcal{Higgs}_X^\infty$ for which the spectral cover $\pi: X_\varphi \rightarrow X$ is totally ramified at x ($\pi^{-1}(x) = ny$).

Theorem 5.5. *Let $(W, p(T))$ be a rational point of $\mathcal{U}_X^\infty \times \mathbb{A}$. Therefore, $(W, p(T))$ lies on the image of $\mathcal{Higgs}_X^\infty(n) \hookrightarrow \mathcal{U}_X^\infty \times \mathbb{A}$ if and only if $p(T)$ is irreducible and:*

$$\sum_{k=0}^{2n-2} \sum_{i+j=k} \text{Res}_{z=0} (\Psi_{W,i}^*(z) \Psi_{\Omega^{-1}}(z) \Psi_{W,j}(z) \text{Tr}(T^{k-1})) \frac{dz}{z^{3g}} = 0.$$

Proof. Using expressions (5.3) and (5.4), Remark 5.7 and Theorem 5.4, we have that $(W, p(T))$ lies in the image of $\mathcal{Higgs}_X^{\infty,n} \hookrightarrow \mathcal{U}_X^\infty \times \mathbb{A}$ if and only if:

$$\text{T}_2 (T \cdot T^{-1+m} \cdot \psi_W^*(z, T, t_n), T^{-1-m} \cdot \psi_{\Omega^{-1}}(z, t) \cdot \psi_W(z, T, t_n) \cdot z^{-3g}) = 0. \quad (5.8)$$

In order to make this formula explicit, observe that the k -algebra structure in V_p is simply the standard polynomial multiplication mod $p(T)$. Thus, by using the polynomial expressions (5.5), (5.6) and the definition of T_2 (see Section 5.2.1), equation (5.8) becomes:

$$\text{Res}_{z=0} \left(\text{Tr} \left(\sum_{k=0}^{2n-2} \sum_{i+j=k} \Psi_{W,i}^*(z) \Psi_{\Omega^{-1}}(z) \Psi_{W,j}(z) T^{k-1} z^{-3g} \right) \right) dz = 0.$$

26 *D. Hernández-Serrano, J. M. Muñoz Porras, F. J. Plaza Martín*

Bearing in mind that the trace map is $k((z))$ -linear and the linearity of Res, we obtain the result. $\square$

Remark 5.9. These techniques can also be used to give a different way of computing the equations for the subscheme $\mathcal{Higgs}_X^\infty(1, \dots, 1)$ (on which the spectral cover is not ramified) addressed in [7].

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References

- [1] A. ÁLVAREZ VÁZQUEZ, *Estructuras aritméticas de las curvas algebraicas*. PhD Thesis, 1996.
- [2] A. ÁLVAREZ VÁZQUEZ, J. M. MUÑOZ PORRAS, AND F. J. PLAZA MARTÍN, *The algebraic formalism of soliton equations over arbitrary base fields*, in Workshop on Abelian Varieties and Theta Functions (Spanish) (Morelia, 1996), vol. 13 of Aportaciones Mat. Investig., Soc. Mat. Mexicana, México, 1998, pp. 3–40.
- [3] A. BEAUVILLE, M. S. NARASIMHAN, AND S. RAMANAN, *Spectral curves and the generalised theta divisor*, J. Reine Angew. Math., 398 (1989), pp. 169–179.
- [4] R. DONAGI AND E. MARKMAN, *Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles*, in Integrable systems and quantum groups (Montecatini Terme, 1993), vol. 1620 of Lecture Notes in Math., Springer, Berlin, 1996, pp. 1–119.
- [5] N. HITCHIN, *Stable bundles and integrable systems*, Duke Math. J., 54 (1987), pp. 91–114.
- [6] ———, *Langlands duality and G_2 spectral curves*, Q. J. Math., 58 (2007), pp. 319–344.
- [7] Y. LI AND M. MULASE, *Hitchin systems and KP equations*, Internat. J. Math., 7 (1996), pp. 227–244.
- [8] I. G. MACDONALD, *Symmetric functions and Hall polynomials*, The Clarendon Press Oxford University Press, New York, 1979. Oxford Mathematical Monographs.
- [9] A. MAFFEI, *The multicomponent KP and Fay trisecant formula*, Internat. Math. Res. Notices, (1996), pp. 769–791.
- [10] M. MULASE, *Category of vector bundles on algebraic curves and infinite-dimensional Grassmannians*, Internat. J. Math., 1 (1990), pp. 293–342.
- [11] M. MULASE AND A. HODGE, *Hitchin integrable systems, deformations of spectral curves and kp-type equations*. Preprint math.AG/0801.0015.v1.
- [12] J. M. MUÑOZ PORRAS AND F. J. PLAZA MARTÍN, *Equations of hurwitz schemes in the infinite grassmannian*. Preprint arXiv:math.AG/0207091v3. To appear in Math. Nachr.
- [13] ———, *Equations of the moduli of pointed curves in the infinite Grassmannian*, J. Differential Geom., 51 (1999), pp. 431–469.
- [14] F. PLAZA MARTÍN, *Relative infinite grassmannian*. Work in progress, 2008.
- [15] M. SATO AND Y. SATO, *Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold*, in Nonlinear partial differential equations in applied science (Tokyo, 1982), vol. 81 of North-Holland Math. Stud., North-Holland, Amsterdam, 1983, pp. 259–271.

- [16] G. SEGAL AND G. WILSON, *Loop groups and equations of KdV type*, Inst. Hautes Études Sci. Publ. Math., (1985), pp. 5–65.
- [17] C. S. SESHADRI, *Fibrés vectoriels sur les courbes algébriques*, vol. 96 of Astérisque, Société Mathématique de France, Paris, 1982. Notes written by J.-M. Drezet from a course at the École Normale Supérieure, June 1980.
- [18] C. T. SIMPSON, *Moduli of representations of the fundamental group of a smooth projective variety. II*, Inst. Hautes Études Sci. Publ. Math., (1994), pp. 5–79 (1995).