



# Geometric approach to Kac–Moody and Virasoro algebras<sup>☆</sup>

E. Gómez González, D. Hernández Serrano<sup>\*</sup>, J.M. Muñoz Porras<sup>\*</sup>, F.J. Plaza Martín<sup>\*</sup>

*Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1–4, 37008 Salamanca, Spain*

*IUFFYM, Instituto Universitario de Física Fundamental y Matemáticas, Universidad de Salamanca, Plaza de la Merced s/n, 37008 Salamanca, Spain*

## ARTICLE INFO

### Article history:

Received 17 March 2011

Received in revised form 12 January 2012

Accepted 1 May 2012

Available online 8 May 2012

### MSC:

primary 14H60

14D21

secondary 22E65

22E67

22E47

### Keywords:

Moduli of vector bundles

Virasoro

Kac–Moody algebra

Infinite Grassmannian

## ABSTRACT

In this paper we show the existence of a group acting infinitesimally transitively on the moduli space of pointed-curves and vector bundles (with formal trivialization data) and whose Lie algebra is an algebra of differential operators. The central extension of this Lie algebra induced by the determinant bundle on the Sato Grassmannian is precisely a semidirect product of a Kac–Moody algebra and the Virasoro algebra. As an application of this geometric approach, we give a local Mumford-type formula in terms of the cocycle associated with this central extension. Finally, using the original Mumford formula we show that this local formula is an infinitesimal version of a general relation in the Picard group of the moduli of vector bundles on a family of curves (without any formal trivialization).

© 2012 Elsevier B.V. All rights reserved.

## 1. Introduction

Uniformization of geometric objects, which is of special mathematical relevance on its own, also has significant consequences in other topics such as mathematical physics. We illustrate this by mentioning a couple of cases. The uniformization theorem of Riemann surfaces was a key issue in Segal's approach to CFT (see the notion of annuli in [1]). Another example is construction of the moduli space of vector bundles on an algebraic curve as a double coset [2], which has been applied in a variety of problems such as proof of the Verlinde formula [3] and development of the geometric theory of conformal blocks [4].

The infinitesimal study of uniformization has also led to connections of moduli theory, integrable systems and representation theory [5,6]. Indeed, if a group is acting on a moduli space such that the action is infinitesimally transitive, then the action of the Lie algebra can help us to the study the properties of that moduli space. Furthermore, a link with the representation theory of infinite Lie algebras also becomes apparent. Arbarello et al. explicitly showed a connection between the representation theory of the Virasoro algebra and the moduli space of line bundles over pointed curves (with formal extra data) [7]. The infinitesimal version of Mumford's formula is another example of this fruitful approach [8].

<sup>☆</sup> This work was partly supported by research contract MTM2009-11393 from Ministerio de Ciencia e Innovación. We would like to thank the referee for valuable comments and suggestions on this paper.

<sup>\*</sup> Corresponding author at: Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1–4, 37008 Salamanca, Spain. Tel.: +34 616314822; fax: +34 923294583.

E-mail addresses: [esteban@usal.es](mailto:esteban@usal.es) (E. Gómez González), [dani@usal.es](mailto:dani@usal.es) (D. Hernández Serrano), [jmp@usal.es](mailto:jmp@usal.es) (J.M. Muñoz Porras), [fplaza@usal.es](mailto:fplaza@usal.es) (F.J. Plaza Martín).

Following this approach, a natural question is to consider the higher rank case, that is, pairs consisting of a curve and a vector bundle (related to the higher rank “generalized”  $bc$ -system [9]). In fact, this higher rank generalization shows a geometric interpretation of a semidirect product of a Kac–Moody algebra and the Virasoro algebra [10,11] and allows us to state a Mumford-type formula for the moduli space of pairs consisting of a curve and a vector bundle.

We explain this in more detail by considering the following. On one hand, using the algebro-geometric version of the Krichever map, we endow the moduli space of the five-tuple  $(X, x, z, E, \phi)$  (consisting of a smooth projective curve  $X$ , a point  $x \in X$ , a local coordinate  $z$  at  $x \in X$ , a rank  $n$  vector bundle  $E$  over  $X$  and a formal trivialization  $\phi$  of  $E$  at  $x$ ) with a scheme structure inside the Sato Grassmannian. On the other hand, we define a group of semilinear automorphisms and show that its Lie algebra is an algebra of first-order differential operators with scalar symbol.

The first relevant fact is that these two objects are related: the group is the uniformization of the moduli space in the sense that its Lie algebra surjects onto the tangent space of the moduli space (Theorem 3.7). Roughly speaking, we could say that this group can be thought of as a “local” moduli space of pairs consisting of a curve and a vector bundle.

The second significant issue is that the central extension induced by the determinant bundle on the Sato Grassmannian of the Lie algebra of the group mentioned above (associated with its action on the Sato Grassmannian) is precisely a semidirect product of a Kac–Moody algebra and the Virasoro algebra. This offers a geometric point of view of this well-known algebra, and it should be noted that it is closely related to Atiyah and  $\mathcal{W}$  algebras, which have appeared in various models of two-dimensional quantum field theory and integrable systems [12–15]. As an application of this geometric reformulation, we explicitly compute the cocycle  $c_{n,\beta}$  associated with the central extension and show the following relation (Theorem 4.4):

$$c_{n,\beta} = \beta c_{n,1} + (1 - \beta)c_{n,0} + 6n\beta(\beta - 1)vir_1, \quad (1.1)$$

where  $vir_1$  is the cocycle associated with the Virasoro algebra. This should be thought of as a higher rank generalization of the infinitesimal version of the Mumford formula [8].

Finally, we prove the following relation (Theorem 5.1) for the Picard group of the moduli of vector bundles on a family of curves (without any formal trivialization):

$$\mathcal{L}_\beta \xrightarrow{\sim} \mathcal{L}_1^{\otimes \beta} \otimes \mathcal{L}_0^{\otimes (1-\beta)} \otimes p^* \lambda_1^{\otimes 6n\beta(\beta-1)},$$

which is a global version of (1.1).

## 2. Moduli of curves with vector bundles

Throughout this paper we use the field  $\mathbb{C}$  of complex numbers, although all results are valid over an arbitrary algebraically closed field of characteristic 0. When no confusion arises, and for the sake of clarity, we deal with rational points (i.e.,  $\mathbb{C}$ -valued points).

We now provide an overview of the relationship between the moduli space of vector bundles and the infinite Sato Grassmannian. The following statements are taken from the literature [16–19].

Let  $\mathcal{U}_g^\infty(n, d)$  denote the moduli functor whose rational points are tuples  $(X, x, z, E, \phi)$ , where  $X$  is a smooth, projective and irreducible curve of genus  $g$  with marking  $x \in X$ ,  $z$  is a formal parameter on  $x$

$$\widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \mathbb{C}[[z]], \quad (2.1)$$

$E$  is a rank- $n$  and degree- $d$  vector bundle on  $X$ , and  $\phi$  is a  $\widehat{\mathcal{O}}_{X,x}$ -module isomorphism

$$\phi: \widehat{E}_x \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}^n. \quad (2.2)$$

Recall that the infinite Grassmannian of  $(V = \mathbb{C}((z))^n, V_+ = \mathbb{C}[[z]]^n)$  has a decomposition into connected components:

$$\mathrm{Gr}(V) = \coprod_{m \in \mathbb{Z}} \mathrm{Gr}^m(V),$$

where  $\mathrm{Gr}^m(V)$  consists of the subspaces  $W \in \mathrm{Gr}(V)$  such that

$$m = \dim_{\mathbb{C}}(W \cap V_+) - \dim_{\mathbb{C}}(V/W + V_+).$$

**Definition 2.1.** The Krichever map is defined by:

$$\begin{aligned} \mathrm{Kr}: \mathcal{U}_g^\infty(n, d) &\longrightarrow \mathrm{Gr}^\chi(V) \\ (X, x, z, E, \phi) &\longmapsto H^0(X - x, E), \end{aligned}$$

where  $H^0(X - x, E)$  is understood as a subspace of  $V$  via the isomorphisms (2.1) and (2.2), and  $\chi = n(1 - g) + d$  (i.e., the Euler–Poincaré characteristic of  $E$ ).

Its image is characterized in the following theorem.

**Theorem 2.2.** *A point  $W \in \text{Gr}^X(V)$  lies on the image of the Krichever map if the stabilizer algebra of  $W$*

$$A_W := \text{Stab}(W) = \{f \in \mathbb{C}((z)) \mid f \cdot W \subseteq W\}$$

*belongs to  $\text{Gr}^{1-g}(\mathbb{C}((z)))$  and  $A_W$  is a regular ring. Moreover, the Krichever morphism is injective.*

**Proof.** The proof follows from [18,19].  $\square$

**Remark 2.3.** Note that the smoothness of  $X$  is equivalent to the regularity of  $A_W$ . If  $X$  is allowed to be singular, then the characterization requires a maximality condition for  $A_W$  analogous to that of [17, Section 6], in which the case of rank 1 was considered. If the aim is to characterize points with values in any  $\mathbb{C}$ -scheme  $S$ , the approach of Mulase can be followed [18]; in this case the flatness of  $A_W$  over  $S$  has to be imposed.

**Theorem 2.4.** *The moduli space  $\mathcal{U}_g^\infty(n, d)$  is representable by a subscheme of  $\text{Gr}^X(V)$ .*

**Proof.** The proof follows from [18,19].  $\square$

We now study this moduli space infinitesimally. With the same notations as before, we introduce  $\mathcal{D}\text{iff}_{X/\mathbb{C}}^1(E, E)$  as the sheaf of differential operators of order  $\leq 1$  from  $E$  to  $E$  over  $\mathcal{O}_X$ . Recall that this fits into the following exact sequence of sheaves:

$$0 \rightarrow \mathcal{E}nd_{\mathcal{O}_X} E \rightarrow \mathcal{D}\text{iff}_{X/\mathbb{C}}^1(E, E) \xrightarrow{\sigma} \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X} E \rightarrow 0,$$

where  $\sigma$  is the symbol morphism [20, Chapter 16] and  $\mathcal{T}_X$  denotes the tangent sheaf.

We consider the subsheaf  $\mathcal{D}_{X/\mathbb{C}}^1(E, E)$  of  $\mathcal{D}\text{iff}_{X/\mathbb{C}}^1(E, E)$  of scalar differential operators, that is, those whose symbol morphism take values in

$$\mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \hookrightarrow \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X} E,$$

where  $\mathcal{O}_X \hookrightarrow \mathcal{E}nd_{\mathcal{O}_X} E$  is the canonical morphism. That is,

$$0 \rightarrow \mathcal{E}nd_{\mathcal{O}_X} E \rightarrow \mathcal{D}_{X/\mathbb{C}}^1(E, E) \xrightarrow{\sigma} \mathcal{T}_X \rightarrow 0. \quad (2.3)$$

Note that if  $E$  has rank 1, then  $\mathcal{D}\text{iff}_{X/\mathbb{C}}^1(E, E) = \mathcal{D}_{X/\mathbb{C}}^1(E, E)$ .

**Remark 2.5.** In the literature, this sequence is also referred to as the Atiyah exact sequence, and  $\mathcal{D}_{X/\mathbb{C}}^1(E, E)$  is the well-known Atiyah bundle [Eq. (3.3) is a formal analog of this sequence]. For the purpose of this paper, it is also interesting to think of this object as the Atiyah algebra of [12]. Furthermore, the bundle  $\mathcal{A}_{\text{Tr} E}$  of [12] is the extension

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{D}_{X/\mathbb{C}}^1(E, E) \xrightarrow{\sigma} \mathcal{T}_X \rightarrow 0$$

induced by the sequence (2.3) and the trace map  $\text{Tr}: \mathcal{E}nd_{\mathcal{O}_X} E \rightarrow \mathcal{O}_X$ .

**Theorem 2.6.** *We have an isomorphism of  $\mathbb{C}$ -vector spaces*

$$\text{T}_{\mathcal{E}} \mathcal{U}_g^\infty(n, d) \xrightarrow{\sim} \varprojlim_m H^1(X, \mathcal{D}_{X/\mathbb{C}}^1(E, E(-mx))),$$

where  $\mathcal{E} = (X, x, z, E, \phi) \in \mathcal{U}_g^\infty(n, d)$  is a rational point.

**Proof.** Let  $\mathcal{U}^m$  be the functor whose rational points are  $\mathcal{E}_m = (X, x, z_m, E, \phi_m)$ , where  $X$  is a smooth projective curve,  $x \in X$  is a point,

$$z_m: \mathcal{O}_X \twoheadrightarrow \mathbb{C}[z]/z^m \mathbb{C}[z]$$

is an  $m$ -level structure at  $x \in X$ ,  $E$  is a rank- $n$  vector bundle on  $X$  and

$$\phi_m: E \twoheadrightarrow (\mathcal{O}_X/\mathcal{O}_X(-mx))^{\oplus n}$$

is an  $m$ -level structure of  $E$  [21]. Note that  $\mathcal{U}_g^\infty(n, d) = \varprojlim_m \mathcal{U}^m$ .

Recalling previous ideas in the literature [7,22–24], we have

$$\text{T}_{\mathcal{E}_m} \mathcal{U}^m \xrightarrow{\sim} H^1(X, \mathcal{D}_{X/\mathbb{C}}^1(E, E(-mx))).$$

Taking the inverse limit in  $m$ , the result follows.  $\square$

**Theorem 2.6** can be also interpreted via the Krichever map. In other words, we can explicitly identify  $T_{\mathcal{E}} \mathcal{U}_g^{\infty}(n, d)$  with a subspace of  $T_{\mathcal{E}} \text{Gr}^X(V)$ . Let  $\mathcal{M}_g^{\infty}$  denote the moduli space of triples  $(X, x, z)$ , where  $X$  has genus  $g$ . We know that [17,25]

$$\begin{aligned} T_W \text{Gr}(V) &\xrightarrow{\sim} \text{Hom}_{\mathbb{C}}(W, V/W) \\ T_{A_W} \mathcal{M}_g^{\infty} &\xrightarrow{\sim} \text{Der}_{\mathbb{C}}(A_W, \mathbb{C}((z))/A_W) \end{aligned}$$

for  $W \in \text{Gr}(V)$  and  $A_W \in \text{Gr}(\mathbb{C}((z)))$ .

**Proposition 2.7.** *Let  $\mathcal{E}$  be a point in  $\mathcal{U}_g^{\infty}(n, d)$  and let  $W$  be its image under the Krichever map. The Krichever map induces an isomorphism of  $\mathbb{C}$ -vector spaces from the tangent space of  $\mathcal{U}_g^{\infty}(n, d)$  at  $\mathcal{E}$  to the space of first-order scalar differential operators from  $W$  to  $V/W$  as  $A_W$ -modules, which is equivalent to*

$$T_{\mathcal{E}} \mathcal{U}_g^{\infty}(n, d) \xrightarrow{\sim} \mathcal{D}_{A_W/\mathbb{C}}^1(W, V/W).$$

**Proof.** Using Theorem 2.2, we can write

$$T_W \mathcal{U}_g^{\infty}(n, d) = \{\overline{W} \in T_W \text{Gr}(V) \text{ such that } A_{\overline{W}} = \text{Stab}(\overline{W}) \in T_{A_W} \text{Gr}(\mathbb{C}((z)))\}.$$

Since  $A_{\overline{W}}$  is a  $\mathbb{C}[\epsilon]/(\epsilon^2)/\epsilon^2$ -algebra, there exists  $g \in \text{Der}_{\mathbb{C}}(A_W, \mathbb{C}((z))/A_W)$  satisfying  $A_{\overline{W}} = \{a + \epsilon g(a), a \in A_W\}$ . By contrast,  $\overline{W} \in T_W \text{Gr}(V)$ , so there exists  $f \in \text{Hom}_{\mathbb{C}}(W, V/W)$  such that

$$\overline{W} = \{w + \epsilon f(w), w \in W\}.$$

Since  $\overline{W}$  is an  $A_{\overline{W}}$ -module, for each  $a \in A_W$  and  $w \in W$  there exists  $w' \in W$  satisfying

$$(a + \epsilon g(a))(w + \epsilon f(w)) = w' + \epsilon f(w'),$$

from where we deduce

$$f(aw) = af(w) + g(a)w.$$

Thus, following [20, Chapter 16],  $f$  belongs to  $\mathcal{D}_{A_W/\mathbb{C}}^1(W, V/W)$ . The converse is straightforward.  $\square$

We now address the dependence between the infinitesimal deformation of the curve and the infinitesimal deformation of the bundle. Let  $\mathcal{U}_g^{\infty}(n, d) \rightarrow \mathcal{M}_g^{\infty}$  be the forgetful functor. Therefore, its fiber at point  $(X, x, z)$  corresponds to  $\mathcal{U}_X^{\infty}(n, d)$ , which denotes the moduli space of pairs consisting of rank- $n$  and degree- $d$  vector bundles over  $X$  endowed with a formal trivialization at  $x \in X$ . Thus, considering the long exact sequence of cohomology of

$$0 \rightarrow \mathcal{H}om_X(E, E(-mx)) \rightarrow \mathcal{D}_{X/\mathbb{C}}^1(E, E(-mx)) \rightarrow \mathcal{T}_X(-mx) \rightarrow 0,$$

we obtains

$$0 \rightarrow T_{(E, \phi)} \mathcal{U}_X^{\infty}(n, d) \rightarrow T_{\mathcal{E}} \mathcal{U}_g^{\infty}(n, d) \rightarrow T_{(X, x, z)} \mathcal{M}_g^{\infty} \rightarrow 0, \quad (2.4)$$

where  $\mathcal{E} = (X, x, z, E, \phi)$ .

In the context of the infinite Grassmannian, (2.4) is (using Proposition 2.7):

$$0 \rightarrow \text{Hom}_{A_W}(W, V/W) \rightarrow \mathcal{D}_{A_W/\mathbb{C}}^1(W, V/W) \rightarrow \text{Der}_{\mathbb{C}}(A_W, \mathbb{C}((z))/A_W) \rightarrow 0.$$

The surjective arrow is the symbol map.

### 3. Uniformization of the moduli space

In this section we define a group of semilinear automorphisms that play the role of “local generator” for the moduli space  $\mathcal{U}_g^{\infty}$ .

**Definition 3.1.** We define the group of automorphisms of  $\mathbb{C}$ -algebras of  $\mathbb{C}((z))$  as the group functor over the category of  $\mathbb{C}$ -schemes given by

$$S \rightsquigarrow G(S) := \text{Aut}_{H^0(S, \mathcal{O}_S) - \text{alg}} H^0(S, \mathcal{O}_S)((z)),$$

where the group law is the composition. This functor is representable by a  $\mathbb{C}$ -scheme that we denote by  $G$  [8].

Let  $V = \mathbb{C}((z))^n$  denote  $n$  copies of the field of the Laurent series and let  $V_+ = \mathbb{C}[[z]]^n$  denote  $n$  copies of the formal power series ring.

**Definition 3.2.** We define the group  $\mathrm{Gl}_{\mathbb{C}}(V)$  as the group of  $\mathbb{C}$ -linear automorphisms  $h: V \xrightarrow{\sim} V$  such that

$$(V_+ + hV_+)/ (V_+ \cap hV_+)$$

is a finite-dimensional  $\mathbb{C}$ -vector space. This group is called the restricted linear group [17,2].

Let  $\mathrm{Gl}_{\mathbb{C}((z))}(V)$  denote the  $\mathbb{C}((z))$ -linear group of  $V = \mathbb{C}((z))^n$ .

**Definition 3.3.** We define the group functor  $\mathrm{SGL}_{\mathbb{C}((z))}(V)$  of semilinear automorphisms of  $V$  as the subfunctor of  $\mathrm{Gl}_{\mathbb{C}}(V)$  whose rational points are  $\mathbb{C}$ -linear automorphisms

$$\gamma: V \xrightarrow{\sim} V$$

for which there exists an automorphism of  $\mathbb{C}$ -algebras of  $\mathbb{C}((z))$ ,  $g \in G$ , satisfying

$$\gamma(z \cdot v) = g(z) \cdot \gamma(v). \quad (3.1)$$

Following previous ideas [8], the definition of this group functor for points with values in any  $\mathbb{C}$ -scheme can be given.

**Proposition 3.4.** We have a canonical exact sequence of group functors

$$0 \rightarrow \mathrm{Gl}_{\mathbb{C}((z))}(V) \rightarrow \mathrm{SGL}_{\mathbb{C}((z))}(V) \rightarrow G \rightarrow 0.$$

Moreover,

$$\mathrm{SGL}_{\mathbb{C}((z))}(V) = \mathrm{Gl}_{\mathbb{C}((z))}(V) \rtimes G.$$

**Proof.** Bearing in mind that  $G$  acts on  $\mathrm{Gl}_{\mathbb{C}((z))}(V)$  by conjugation, the result follows from [26, Chapter IV.6]. In particular, the composition law in  $\mathrm{SGL}_{\mathbb{C}((z))}(V)$  is explicitly given by

$$(\gamma_1, g_1) * (\gamma_2, g_2) = (c_{g_2}(\gamma_1) \circ \gamma_2, g_1 \circ g_2),$$

where  $c_{g_2}(\gamma_1) = g_2^{-1} \circ \gamma_1 \circ g_2$  is the action of  $G$  on  $\mathrm{Gl}_{\mathbb{C}((z))}(V)$  by conjugation.  $\square$

Let  $\mathfrak{g}$ ,  $\mathfrak{gl}_{\mathbb{C}((z))}(V)$  and  $\mathfrak{sgl}_{\mathbb{C}((z))}(V)$  denote the Lie algebras of  $G$ ,  $\mathrm{Gl}_{\mathbb{C}((z))}(V)$  and  $\mathrm{SGL}_{\mathbb{C}((z))}(V)$ , respectively. Let  $\mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V, V)$  denote the space of differential operators of order  $\leq 1$  from  $V$  to  $V$  over  $\mathbb{C}((z))$ , and consider the subspace  $\mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V, V)$  of scalar differential operators [20, Chapter 16].

**Proposition 3.5.**  $\mathfrak{sgl}_{\mathbb{C}((z))}(V) = \mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V, V)$  are Lie subalgebras of  $\mathrm{End}_{\mathbb{C}} V$ .

**Proof.** Let  $\mathbb{C}[\epsilon]/(\epsilon^2)$  be the ring of dual numbers. By definition,  $\mathfrak{sgl}_{\mathbb{C}((z))}(V)$  consists of  $\mathbb{C}[\epsilon]/(\epsilon^2)$ -linear automorphisms  $\gamma$  of  $V \oplus \epsilon V$  such that  $\gamma|_{\epsilon=0} = \mathrm{Id}$  and for which there exists a  $\mathbb{C}[\epsilon]/(\epsilon^2)$ -algebra automorphism

$$g: \mathbb{C}((z)) \oplus \epsilon \mathbb{C}((z)) \xrightarrow{\sim} \mathbb{C}((z)) \oplus \epsilon \mathbb{C}((z))$$

satisfying  $g|_{\epsilon=0} = \mathrm{Id}$  and  $\gamma(zv) = g(z)\gamma(v)$ .

Since  $\gamma$  is a  $\mathbb{C}[\epsilon]/(\epsilon^2)$ -linear automorphism, we can write  $\gamma = \mathrm{Id} + \epsilon \gamma_0$ , where  $\gamma_0 \in \mathrm{End}_{\mathbb{C}} V$ . Similarly, since  $g$  is a  $\mathbb{C}[\epsilon]/(\epsilon^2)$ -algebra automorphism, this implies that  $g = 1 + \epsilon g_0$ , where  $g_0 \in \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}_{\mathbb{C}}(\mathbb{C}((z)))$  [8]. The condition

$$(\mathrm{Id} + \epsilon \gamma_0)(zv) = (1 + \epsilon g_0)(z)(\mathrm{Id} + \epsilon \gamma_0)(v)$$

implies

$$\gamma_0(zv) = z\gamma_0(v) + g_0(z)v, \quad (3.2)$$

that is,  $\gamma_0 \in \mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V, V)$ . Thus, we have obtained a  $\mathbb{C}$ -vector space isomorphism

$$\begin{aligned} \mathfrak{sgl}_{\mathbb{C}((z))}(V) &\xrightarrow{\sim} \mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V, V) \\ \mathrm{Id} + \epsilon \gamma_0 &\mapsto \gamma_0. \end{aligned}$$

It remains to show that this is a Lie algebra isomorphism. Observe that by its very definition,  $\mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V, V)$  fits into the following exact sequence:

$$0 \rightarrow \mathrm{End}_{\mathbb{C}((z))} V \rightarrow \mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V, V) \xrightarrow{\sigma} \mathrm{Der}_{\mathbb{C}}(\mathbb{C}((z))) \rightarrow 0, \quad (3.3)$$

where  $\sigma$  is the symbol map. Note that  $\sigma(\gamma_0) = g_0$ .

Bearing in mind that  $\text{Der}_{\mathbb{C}}(\mathbb{C}((z))) = \mathbb{C}((z))\partial_z$  and that this sequence splits as a sequence of vector spaces, we can express the elements of  $\mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V, V)$  as  $\gamma + g\partial_z$ , where  $\gamma \in \text{End}_{\mathbb{C}((z))} V$  and  $g \in \text{Der}_{\mathbb{C}}(\mathbb{C}((z)))$ . However, the Lie bracket of  $\mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V, V)$ , which is canonically inherited from that of  $\text{End}_{\mathbb{C}} V$ , is given by:

$$[\gamma_1 + g_1\partial_z, \gamma_2 + g_2\partial_z] = [\gamma_1, \gamma_2] + [g_1\partial_z, g_2\partial_z] + g_1\partial_z(\gamma_2) - g_2\partial_z(\gamma_1).$$

Set  $\{L_r = z^{r+1}\partial_z \mid r \in \mathbb{Z}\}$  as a basis for  $\text{Der}_{\mathbb{C}}(\mathbb{C}((z)))$  and  $\{E_{ij}^s \mid s \in \mathbb{Z}, i, j = 1, \dots, n\}$  as a basis for  $\text{End}_{\mathbb{C}((z))} V$  (where  $E_{ij}^s$  is an  $n \times n$  matrix whose  $(i, j)$  entry is  $z^s$  and 0 otherwise). The Lie bracket of  $\mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V, V)$  is given by the following rules [27]:

$$\begin{aligned} [L_r, L_s] &= (s - r)L_{r+s} \\ [E_{ij}^r, E_{kl}^s] &= \delta_{jk}E_{il}^{r+s} - \delta_{li}E_{kj}^{s+r} \\ [L_r, E_{kl}^s] &= sE_{kl}^{r+s}. \end{aligned} \quad (3.4)$$

It is straightforward to check that these expressions coincide with those for the Lie bracket of  $\mathfrak{sl}_{\mathbb{C}((z))}(V)$ .  $\square$

**Remark 3.6.** Note that (3.2) means that  $\gamma_0$  can be viewed as a covariant derivative along the vector field  $g_0(z) \in \text{Der}_{\mathbb{C}}(\mathbb{C}((z)))$ . This type of structure is also considered in [14, Section 1].

We now show that the group  $\text{SGL}_{\mathbb{C}((z))}(V)$  can be thought of as a “local generator” for the moduli space  $\mathcal{U}_g^\infty(n, d)$ .

**Theorem 3.7.** *The group  $\text{SGL}_{\mathbb{C}((z))}(V)$  acts on  $\mathcal{U}_g^\infty(n, d)$  and this action is locally transitive.*

**Proof.** First, note that the action of  $\text{SGL}_{\mathbb{C}((z))}(V)$  on  $V$  gives rise to an action of  $\text{SGL}_{\mathbb{C}((z))}(V)$  on  $\text{Gr}^X(V)$ . Recall (Theorem 2.2) that rational points of  $\mathcal{U}_g^\infty(n, d)$  correspond to the points  $W \in \text{Gr}^X(V)$  such that  $A_W$  belongs to  $\text{Gr}^{1-g}(\mathbb{C}((z)))$  and  $A_W$  is regular.

We must first check that  $\text{SGL}_{\mathbb{C}((z))}(V)$  acts on  $\mathcal{U}_g^\infty(n, d)$ ; this means that for all  $\gamma \in \text{SGL}_{\mathbb{C}((z))}(V)$  and all  $W \in \mathcal{U}_g^\infty(n, d)$ ,  $\gamma(W) \in \text{Gr}^X(V)$  and  $A_{\gamma(W)} \in \text{Gr}^{1-g}(\mathbb{C}((z)))$  hold. It is straightforward to see that  $\gamma(W) \in \text{Gr}^X(V)$ . Moreover,

$$\begin{aligned} A_{\gamma(W)} &= \{a \in \mathbb{C}((z)) \mid a \cdot \gamma(W) \subseteq \gamma(W)\} = \{a \in \mathbb{C}((z)) \mid \gamma^{-1}(a \cdot \gamma(W)) \subseteq W\} \\ &= \{a \in \mathbb{C}((z)) \mid g(a) \cdot W \subseteq W\} = g(A_W), \end{aligned}$$

where  $g \in G$  satisfies  $\gamma(zv) = g(z)\gamma(v)$  (Definition 3.3). The conclusion follows since  $g(A_W) \in \text{Gr}^{1-g}(\mathbb{C}((z)))$  [8, Theorem 4.9] and  $g(A_W)$  is regular.

To see that the action of  $\text{SGL}_{\mathbb{C}((z))}(V)$  is locally transitive, it suffices to prove that the orbit morphism is surjective at the level of tangent spaces [8]. That is, we have to check that

$$\mathfrak{sl}_{\mathbb{C}((z))}(V) \rightarrow T_{\mathcal{U}_g^\infty(n, d)}$$

is surjective for all  $\mathcal{E} = (X, x, z, E, \phi) \in \mathcal{U}_g^\infty(n, d)$ . Consider the following exact sequence:

$$\mathcal{D}_{X/k}^1(E, E(-mx)) \hookrightarrow \mathcal{D}_{X/k}^1(E, E(\bar{m}x)) \rightarrow \mathcal{D}_{X/k}^1(E, (\mathcal{O}_X(\bar{m}x)/\mathcal{O}_X(-mx))^n).$$

Taking cohomology  $\varprojlim_m$  and  $\varinjlim_m$ , we obtain

$$\begin{aligned} 0 \rightarrow H^0(X - x, \mathcal{D}_{X/\mathbb{C}}^1(E, E)) &\rightarrow \varinjlim_{\bar{m}} \varprojlim_m \mathcal{D}_{X/k}^1(E, (\mathcal{O}_X(\bar{m}x)/\mathcal{O}_X(-mx))^n) \\ &\rightarrow \varprojlim_m H^1(X, \mathcal{D}_{X/\mathbb{C}}^1(E, E(-mx))) \rightarrow 0, \end{aligned} \quad (3.5)$$

where  $\varprojlim_m H^1(X, \mathcal{D}_{X/\mathbb{C}}^1(E, E(-mx))) \xrightarrow{\sim} T_{\mathcal{U}_g^\infty(n, d)}$  by Theorem 2.6.

Note that

$$\varinjlim_{\bar{m}} \varprojlim_m \mathcal{D}_{X/k}^1(E, (\mathcal{O}_X(\bar{m}x)/\mathcal{O}_X(-mx))^n) = \mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V, V) = \mathfrak{sl}_{\mathbb{C}((z))}(V)$$

implies the exactness of the sequence

$$0 \longrightarrow H^0(X - x, \mathcal{D}_{X/\mathbb{C}}^1(E, E)) \longrightarrow \mathfrak{sl}_{\mathbb{C}((z))}(V) \longrightarrow T_{\mathcal{U}_g^\infty(n, d)} \longrightarrow 0$$

and the statement is proved.  $\square$

**Remark 3.8.** There are similar sequences for  $\mathcal{M}_g^\infty$  and  $\mathcal{U}_X^\infty(n, d)$  to that for  $\mathcal{U}_g^\infty(n, d)$  given in (3.5) [8]. Therefore, we have the following diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & H^0(X - x, \mathcal{E}nd_X E) & \longrightarrow & \mathfrak{gl}_{\mathbb{C}((z))}(V) & \longrightarrow & T_{(E, \phi)} \mathcal{U}_X^\infty(n, d) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H^0(X - x, \mathcal{D}_{X/\mathbb{C}}^1(E, E)) & \longrightarrow & \mathfrak{sgl}_{\mathbb{C}((z))}(V) & \longrightarrow & T_\varepsilon \mathcal{U}_g^\infty(n, d) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H^0(X - x, \mathcal{T}_X) & \longrightarrow & \mathfrak{g} & \longrightarrow & T_{(X, x, z)} \mathcal{M}_g^\infty \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

which is identified (via the Krichever map) with the diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \text{End}_{A_W} W & \longrightarrow & \text{End}_{\mathbb{C}((z))} V & \longrightarrow & \text{Hom}_{A_W}(W, V/W) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathcal{D}_{A_W/\mathbb{C}}^1(W) & \longrightarrow & \mathcal{D}_{\mathbb{C}((z))/\mathbb{C}}^1(V) & \longrightarrow & \mathcal{D}_{A_W/\mathbb{C}}^1(W, V/W) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Der}_{\mathbb{C}}(A_W) & \longrightarrow & \text{Der}_{\mathbb{C}}(\mathbb{C}((z))) & \longrightarrow & \text{Der}_{\mathbb{C}}(A_W, \mathbb{C}((z))/A_W) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

**Remark 3.9.** Our approach can be viewed as a generalization of previous works [7,25]. In fact, if we restrict ourselves to the rank 1 case and look at the level of tangent spaces, then the relation to the results of Arbarello et al. [7] becomes apparent. Furthermore, here we are essentially studying bundles on curves, whereas Muñoz Porras and Plaza Martín [25] deal with coverings of curves; the techniques and goals are, however, similar and closely related.

**Remark 3.10.** Falqui and Reina used similar techniques to prove that for  $n = 2, d = 1$ , there is an isomorphism between the second cohomology group of  $\mathcal{D}^1 = \mathfrak{sl}_{\mathbb{C}((z))}(V) \rtimes \mathfrak{g}$  (where  $\mathfrak{sl}_{\mathbb{C}((z))}(V)$  denotes the Lie algebra of the special linear group) and the second rational cohomology group of the moduli space parametrizing four-tuples consisting of a curve, a point, a tangent vector and a rank-2 vector bundle with fixed determinant [28]. This cohomological application for higher rank and arbitrary determinants will be the subject of future research.

#### 4. Central extensions and cocycles: Kac–Moody and Virasoro algebras and a local Mumford-type formula

In this section we demonstrate that some central extensions defined by the Lie algebra  $\mathfrak{sgl}_{\mathbb{C}((z))}(V)$  come from pullbacks of algebras of the Virasoro type, while others arise as intertwinements of Kac–Moody and Virasoro algebras. We explicitly compute the associated cocycles and give a relation among them that should be thought of as a local Mumford-type formula.

First, we review some facts concerning the construction of a family of Virasoro algebras [8, Section 3.5]. Let  $\text{Gr}(V)$  denote the infinite Grassmannian associated with  $(V, V_+)$  [16,17]. Its rational points correspond to the vector subspaces  $W \subseteq V$  such that

$$W \cap V_+ \text{ and } V/W + V_+$$

are finite-dimensional vector spaces over  $\mathbb{C}$ .

The group  $\text{GL}_{\mathbb{C}}(V)$  acts on  $\text{Gr}(V)$  and preserves the determinant bundle. Therefore, we have the canonical central extension induced by the determinant bundle

$$1 \rightarrow \mathbb{C}^* \rightarrow \widehat{\text{GL}}_{\mathbb{C}}(V) \rightarrow \text{GL}_{\mathbb{C}}(V) \rightarrow 1$$

and the cocycle associated with this central extension is given by

$$c(g_1, g_2) = \det(\bar{g}_1 \circ (\bar{g}_1 \circ \bar{g}_2)^{-1} \circ \bar{g}_2),$$

where  $\bar{g}_i$  are preimages of  $g_i$ .

Let  $\text{Id} + \epsilon_i D_i$  be a  $\mathbb{C}[\epsilon_i]/\epsilon_i^2$ -valued point of  $\text{Gl}_{\mathbb{C}}(V)$ . The very definition of the cocycle at the Lie algebra level yields the expression

$$c_{\text{Lie}}(D_1, D_2) = \text{Tr}(D_1^{+-} D_2^{-+} - D_2^{+-} D_1^{-+}), \quad (4.1)$$

where  $D_i^{+-}: V^+ \rightarrow V^-$  is induced by  $\text{Id} + \epsilon_i D_i$  w.r.t. the decomposition  $V \xrightarrow{\sim} V^- \oplus V^+$  where  $V^- = z^{-1}\mathbb{C}[z^{-1}]^n$ ,  $V^+ = \mathbb{C}[[z]]^n$ .

Fix an integer number  $\beta$  and consider the  $\mathbb{C}$ -vector space  $\mathbb{C}((z))(dz)^{\otimes \beta}$ . There is an action of  $G$  on  $\mathbb{C}((z))(dz)^{\otimes \beta}$  defined by

$$\begin{aligned} \mu_{\beta}: G \times \mathbb{C}((z))(dz)^{\otimes \beta} &\rightarrow \mathbb{C}((z))(dz)^{\otimes \beta} \\ (g(z), f(z)(dz)^{\otimes \beta}) &\mapsto f(g(z))g'(z)^{\beta}(dz)^{\otimes \beta}, \end{aligned} \quad (4.2)$$

which induces an action on  $\text{Gr}(\mathbb{C}((z))(dz)^{\otimes \beta})$  (denoted again by  $\mu_{\beta}$ ) verifying

$$\mu_{\beta}(g(z)) = g'(z)^{\beta} \mu_0(g(z))$$

and preserving the determinant line bundle. Therefore, we may consider the associated central extension

$$1 \rightarrow \mathbb{C}^* \rightarrow \widehat{G}_{\beta} \rightarrow G \rightarrow 1. \quad (4.3)$$

As  $\beta$  varies in  $\mathbb{Z}$ , the cocycles corresponding to these central extensions are as follows:

$$\text{vir}_{\beta}(L_r, L_s) = \delta_{r,-s} \left( \frac{r^3 - r}{6} \right) (1 - 6\beta + 6\beta^2) \quad (4.4)$$

at the Lie algebra level. Note that  $\widehat{\mathfrak{g}}_1$  is precisely the Virasoro algebra, and the formula

$$\text{vir}_{\beta} = \text{vir}_1 \cdot (1 - 6\beta + 6\beta^2) \quad (4.5)$$

is a local analog of the Mumford formula, where  $\text{vir}_1$  is the standard cocycle associated with the Virasoro algebra.

**Remark 4.1.** Note that  $\mathcal{M}_g^{\infty}$  is stable under the action of  $G$  [8] and an explicit analytic description of this action has been reported by Biswas and Raina [29].

Following these ideas, we compute new cocycles. Note that each central extension

$$1 \rightarrow \mathbb{C}^* \rightarrow \widehat{G}_{\beta} \rightarrow G \rightarrow 1$$

can be pulled back to  $\text{SGL}_{\mathbb{C}((z))}(V)$  by the surjection  $\text{SGL}_{\mathbb{C}((z))}(V) \xrightarrow{p_n} G$  (recall that  $n = \dim_{\mathbb{C}((z))} V$ ), yielding a central extension

$$1 \rightarrow \mathbb{C}^* \rightarrow \text{SGL}_{\mathbb{C}((z))}(V) \times_G \widehat{G}_{\beta} \rightarrow \text{SGL}_{\mathbb{C}((z))}(V) \rightarrow 1. \quad (4.6)$$

Let  $\text{vir}_{n,\beta}$  denote the cocycle corresponding to this central extension induced at the Lie algebra level. By construction, we have

$$\text{vir}_{n,\beta} = p_n^*(\text{vir}_{\beta}). \quad (4.7)$$

More explicitly, the following formulae hold:

$$\begin{aligned} \text{vir}_{n,\beta}(L_r, L_s) &= n \cdot \delta_{r,-s} \cdot \frac{s^3 - s}{6} (1 - 6\beta + 6\beta^2) \\ \text{vir}_{n,\beta}(E_{ij}^r, E_{kl}^s) &= 0 \\ \text{vir}_{n,\beta}(L_r, E_{ij}^s) &= 0 \end{aligned} \quad (4.8)$$

with the same notations as for (3.4). In particular,  $\text{vir}_{n,\beta} = n \cdot \text{vir}_{1,\beta}$ .

The next step is to find a family of central extensions of  $\text{SGL}_{\mathbb{C}((z))}(V)$  that intertwine the structure of both  $\mathfrak{gl}_{\mathbb{C}((z))}(V)$  Kac–Moody and Virasoro algebras at the Lie algebra level.

Similarly to the case of  $G$ , there is a natural action of  $\text{SGL}_{\mathbb{C}((z))}(V)$  on  $V_{n,\beta} = (\mathbb{C}((z))(dz)^{\otimes \beta})^n$  defined by

$$\mu_{n,\beta}(\gamma(z \cdot v)) = g'(z)^{\beta} \cdot \gamma(v).$$

**Theorem 4.2.** The action  $\mu_{n,\beta}$  induces an action of  $\text{SGL}_{\mathbb{C}((z))}(V)$  on  $\text{Gr}(V)$  that preserves the determinant line bundle, and therefore there exists a central extension

$$1 \rightarrow \mathbb{C}^* \rightarrow \widehat{\text{SGL}}_{\mathbb{C}((z))}^{\beta}(V) \rightarrow \text{SGL}_{\mathbb{C}((z))}(V) \rightarrow 1$$

canonically associated with  $\mu_{n,\beta}$ .

**Proof.** By Proposition 3.4,  $\text{SGL}_{\mathbb{C}((z))}(V)$  is the subgroup of  $\text{GL}_{\mathbb{C}}(V)$  generated by  $G$  and  $\text{GL}_{\mathbb{C}((z))}(V)$ . Using [8, Theorem 2.2], the first part of the statement follows. The existence of the central extension is a consequence of [8, Theorem 2.3].  $\square$

Taking into account the expressions of Eq. (3.4), the Lie algebra structure of  $\widehat{\text{sgl}}_{\mathbb{C}((z))}^{\beta}(V)$  is governed by the following rules:

$$\begin{aligned}[L_r, L_s] &= (s - r)L_{r+s} + c_{n,\beta}(L_r, L_s) \\ [E_{ij}^r, E_{kl}^s] &= \delta_{jk}E_{il}^{r+s} - \delta_{il}E_{kj}^{r+s} + c_{n,\beta}(E_{ij}^r, E_{kl}^s) \\ [L_r, E_{kl}^s] &= sE_{kl}^{r+s} + c_{n,\beta}(L_r, E_{kl}^s),\end{aligned}$$

where  $c_{n,\beta}$  denotes the corresponding cocycle. It is then clear that  $\widehat{\text{sgl}}_{\mathbb{C}((z))}^{\beta}(V)$  is a family of a semidirect product of a  $\text{gl}_{\mathbb{C}((z))}(V)$  Kac–Moody and a Virasoro algebra.

**Proposition 4.3.** The cocycle  $c_{n,\beta}$  associated with the central extension  $\widehat{\text{sgl}}_{\mathbb{C}((z))}^{\beta}(V)$  is given by

$$\begin{aligned}c_{n,\beta}(L_r, L_s) &= n \cdot \delta_{r,-s} \cdot \frac{r^3 - r}{6} (1 - 6\beta + 6\beta^2) \\ c_{n,\beta}(E_{ij}^r, E_{kl}^s) &= \delta_{r,-s} \delta_{il} \delta_{jk} \cdot s \\ c_{n,\beta}(L_r, E_{ij}^s) &= \delta_{r,-s} \delta_{ij} \cdot \frac{r(r+1)}{2} (1 - 2\beta).\end{aligned}$$

**Proof.** Let  $\{e_k \mid k \in \mathbb{Z}\}$  be a basis of  $V$ , where  $e_k = (0, \dots, 0, z_{k_2}^{k_1}, 0, \dots, 0)$  (i.e.  $z^{k_1}$  in the  $k_2$ -th entry and 0 elsewhere) and  $k = k_1 n + k_2 - 1$  with  $k_1 \in \mathbb{Z}$  and  $k_2 = 1, \dots, n$ .

Recall that the action of  $\text{SGL}_{\mathbb{C}((z))}(V)$  on  $V_{n,\beta} = (\mathbb{C}((z))(dz)^{\otimes \beta})^n$  is defined by

$$\mu_{n,\beta}(\gamma(z \cdot v)) = g'(z)^{\beta} \cdot \gamma(v).$$

Therefore, the action of  $\text{GL}_{\mathbb{C}((z))}(V)$  does not depend on  $\beta$ . The  $\mathbb{Z} \times \mathbb{Z}$ -matrix associated with  $\mu_{n,\beta}(L_r)$  is

$$(\mu_{n,\beta}(L_r))_{lk} = \begin{cases} 1 & \text{if } l_1 = k_1 \text{ and } l_2 = k_2 \\ \epsilon \cdot (k_1 + \beta(1+r)) & \text{if } l_1 = k_1 + r \text{ and } l_2 = k_2 \\ 0 & \text{otherwise,} \end{cases}$$

and the element in  $\text{End}_{\mathbb{C}}(V)$  corresponding to  $(\text{Id} + \epsilon E_{ij}^r)$  is

$$(\text{Id} + \epsilon E_{ij}^r)_{lk} = \begin{cases} 1 & \text{if } l_1 = k_1 \text{ and } l_2 = k_2 \\ \epsilon & \text{if } l_1 = k_1 + r \text{ and } l_2 = i \text{ and } k_2 = j \\ 0 & \text{otherwise.} \end{cases}$$

Taking into account (4.1), the result follows.  $\square$

Using the above results, we can check that the semidirect product of the Virasoro and Kac–Moody algebra considered by Goddard and Olive [11] (quantum physics) coincides with the Lie algebra  $\widehat{\text{sgl}}_{\mathbb{C}((z))}^{\beta}(V)$  for  $\beta = \frac{1}{2}$ . See also Eq. (4.2) of Semikhatov for the rank 1 case [30].

**Theorem 4.4** (Local Mumford-type Formula). The following relation holds:

$$c_{n,\beta} = \beta c_{n,1} + (1 - \beta)c_{n,0} + 6n\beta(\beta - 1)\text{vir}_1.$$

**Proof.** Using Proposition 4.3 and Eq. (4.8), the values of the cocycles for a pair of elements of the basis can be arranged as in the following table.

|                    | $(L_r, L_s)$                                                      | $(L_r, E_{ij}^s)$                                               | $(E_{ij}^r, E_{kl}^s)$                          |
|--------------------|-------------------------------------------------------------------|-----------------------------------------------------------------|-------------------------------------------------|
| $\text{vir}_{n,1}$ | $n \cdot \delta_{r,-s} \frac{r^3 - r}{6}$                         | 0                                                               | 0                                               |
| $c_{n,0}$          | $n \cdot \delta_{r,-s} \frac{r^3 - r}{6}$                         | $\delta_{r,-s} \delta_{ij} \cdot \frac{r(r+1)}{2}$              | $\delta_{r,-s} \delta_{il} \delta_{jk} \cdot s$ |
| $c_{n,1}$          | $n \cdot \delta_{r,-s} \frac{r^3 - r}{6}$                         | $-\delta_{r,-s} \delta_{ij} \cdot \frac{r(r+1)}{2}$             | $\delta_{r,-s} \delta_{il} \delta_{jk} \cdot s$ |
| $c_{n,\beta}$      | $n \cdot \delta_{r,-s} \frac{r^3 - r}{6} (1 - 6\beta + 6\beta^2)$ | $\delta_{r,-s} \delta_{ij} \cdot \frac{r(r+1)}{2} (1 - 2\beta)$ | $\delta_{r,-s} \delta_{il} \delta_{jk} \cdot s$ |

From the table it is evident that

$$c_{n,\beta} = \beta c_{n,1} + (1 - \beta)c_{n,0} + 6\beta(\beta - 1)\text{vir}_{n,1}$$

and by virtue of (4.7) we have

$$\text{vir}_{n,1} = n \cdot \text{vir}_1.$$

Therefore,

$$c_{n,\beta} = \beta c_{n,1} + (1 - \beta)c_{n,0} + 6n\beta(\beta - 1)\text{vir}_1. \quad \square$$

We now show how the above result can be restated in terms of bitorsors or, equivalently, line bundles over  $\text{SGL}_{\mathbb{C}((z))}(V)$  (see [31, Exposé VII] for the relationships among bitorsors, line bundles and extensions). Let  $\mathbb{L}_{n,\beta}$  denote the bitorsor over  $\text{SGL}_{\mathbb{C}((z))}(V)$  associated with  $\widehat{\text{SG}}_{\mathbb{C}((z))}^\beta(V)$  (Theorem 4.2) and let  $\Lambda_1$  be the bitorsor over  $G$  corresponding to  $\widehat{G}_1$  [Eq. (4.3)]. Thus, Theorem 4.4 is equivalent to the following identity:

$$\mathbb{L}_{n,\beta} \simeq \mathbb{L}_{n,1}^{\otimes \beta} \otimes \mathbb{L}_{n,0}^{\otimes (1-\beta)} \otimes p^* \Lambda_1^{\otimes 6n\beta(\beta-1)}, \quad (4.9)$$

where  $p$  is the natural projection map  $\text{SGL}_{\mathbb{C}((z))}(V) \rightarrow G$ .

We finish this section with a brief discussion of the properties of  $\text{SGL}_{\mathbb{C}((z))}(V)$  and, more precisely, of the relationship between its Lie algebra and the Atiyah and  $\mathcal{W}_{1+\infty}$  algebras. From our point of view, these connections make  $\text{SGL}_{\mathbb{C}((z))}(V)$  a relevant object that deserves deeper study.

First, observe that there is a pullback map from the central extensions of  $\mathfrak{sgl}_{\mathbb{C}((z))}(V)$  to those of  $\mathfrak{gl}_{\mathbb{C}((z))}(V)$

$$H^2(\mathfrak{sgl}_{\mathbb{C}((z))}(V), \mathbb{C}) \longrightarrow H^2(\mathfrak{gl}_{\mathbb{C}((z))}(V), \mathbb{C}) \quad (4.10)$$

defined by restriction through the inclusion  $\mathfrak{gl}_{\mathbb{C}((z))}(V) \hookrightarrow \mathfrak{sgl}_{\mathbb{C}((z))}(V)$ . Note that the Kac–Moody algebra is a central extension of  $\mathfrak{gl}_{\mathbb{C}((z))}(V)$ . In this way, we establish a link with the theory of Kac–Moody algebras and by considering the action on the spaces of global sections of powers of the determinant bundle, we obtain semi-infinite wedge representations of such algebras [27, Lecture 9]. In addition, recall that Section 4 provides a map

$$H^2(\mathfrak{g}, \mathbb{C}) \hookrightarrow H^2(\mathfrak{sgl}_{\mathbb{C}((z))}(V), \mathbb{C}),$$

where  $\mathfrak{g} = \text{Lie}(G)$  is referred to in the literature as the Witt algebra and its central extension is the Virasoro algebra.

Summing up, the structures of both Kac–Moody and Virasoro algebras are intertwined naturally into a single object, namely, the group functor  $\text{SGL}_{\mathbb{C}((z))}(V)$ . This group is endowed with a geometric meaning (in terms of vectors bundles over algebraic curves) in the following section.

Furthermore, the Lie algebra  $\mathfrak{sgl}_{\mathbb{C}((z))}(V)$  can be thought of as a formal analog to the so-called Atiyah algebras [12]. It is worth pointing out that by the equivalence of categories between Atiyah algebras and algebras of differential operators [12, Section 1], the algebra of differential operators associated with  $\mathfrak{sgl}_{\mathbb{C}((z))}(V)$  is precisely  $\mathfrak{gl}_{\mathbb{C}((z))}(V) \otimes_{\mathbb{C}} \mathbb{C}[[\partial_z]]$ ; that is, the algebra of differential operators of arbitrary order whose coefficients are matrices.

It is well known [32] that  $\dim H^2(\mathfrak{gl}_{\mathbb{C}((z))}(V) \otimes_{\mathbb{C}} \mathbb{C}[[\partial_z]], \mathbb{C}) = 1$  or, in other words, that it has essentially a unique central extension (the simplest case dates back to 1989 [33]). Bearing in mind the explicit expressions for such 2-cocycles in the literature [34,27], we can describe such an extension explicitly for the case of  $\mathfrak{gl}_{\mathbb{C}((z))}(V) \otimes_{\mathbb{C}} \mathbb{C}[[\partial_z]]$ :

$$\Psi(A(z)\partial_z^r, B(z)\partial_z^s) := \frac{r!s!}{(r+s+1)!} \text{Res}_{z=0} \text{Tr}(\partial_z^{s+1}A(z) \cdot \partial_z^r B(z))dz.$$

It is remarkable that restriction of this cocycle to the Lie subalgebra of differential operators of order  $\leq 1$  coincides with the expression computed in Proposition 4.3 for  $\beta = 0$ .

We now focus on the case of  $\dim_{\mathbb{C}((z))} V = 1$ , that is,  $V = \mathbb{C}((z))$ . The Lie algebra corresponding to the above cocycle is called the  $\mathcal{W}_{1+\infty}$ -algebra and its representation theory in terms of vertex operator algebras has been studied in depth [13,15,35]. To shed some light on this, note that the action on the Fock space

$$\widehat{\text{SGL}}_{\mathbb{C}((z))}^1(\mathbb{C}((z))) \hookrightarrow \text{GL}(H^0(\text{Gr}(\mathbb{C}((z))), \text{Det}^*))$$

induces a map between their Lie algebras that extends to

$$\mathcal{W}_{1+\infty} \longrightarrow \text{End}(H^0(\text{Gr}(\mathbb{C}((z))), \text{Det}^*)).$$

In this case, the vector space parametrizing central extensions of the Lie algebra  $\mathfrak{sgl}_{\mathbb{C}((z))}(\mathbb{C}((z)))$  was explicitly computed and applied to compute some cohomology groups of moduli spaces [7]. The authors showed that this space is three-dimensional and is generated by the following 2-cocycles:

$$\begin{aligned} \alpha_1(f_1\partial_z + g_1, f_2\partial_z + g_2) &= \text{Res}_{z=0} f_1\partial_z^3 f_2 dz \\ \alpha_2(f_1\partial_z + g_1, f_2\partial_z + g_2) &= \text{Res}_{z=0} (f_1\partial_z^2 g_2 - f_2\partial_z^2 g_1) dz \\ \alpha_3(f_1\partial_z + g_1, f_2\partial_z + g_2) &= \text{Res}_{z=0} g_1\partial_z g_2 dz. \end{aligned}$$

The Lie algebra  $\mathfrak{sgl}_{\mathbb{C}((z))}(\mathbb{C}((z)))$  has also been presented as the algebra of asymptotic symmetries of warped black-hole geometries [36, Eq. (16)].

Unfortunately, when  $V$  is of arbitrary dimension, the whole group is not known. However, Theorem 4.4 shows that

$$\text{vir}_{n,\beta} \in \langle \text{vir}_{n,1}, \text{vir}_{n,0}, p_n^*(\text{vir}) \rangle \subseteq H^2(\mathfrak{sgl}_{\mathbb{C}((z))}(V), \mathbb{C})$$

and provides the coefficients of the linear combination. Further research will be carried out show that  $H^2(\mathfrak{sgl}_{\mathbb{C}((z))}(V), \mathbb{C}) \xrightarrow{\sim} \mathbb{C}^3$  and to generalize (to higher rank case) some of the cohomological results of Arbarello et al. [7].

Finally, interested readers are referred to the work of Schlichenmaier [37] for a cohomological study of central extensions of Krichever–Novikov algebras, which in a certain sense generalize the case of  $\mathfrak{sl}_{\mathbb{C}((z))}(V)$ .

## 5. A relation in the Picard group of moduli of vector bundles for a family of curves

In this section we show that Eq. (4.9) is of geometric nature. We show that certain line bundles on the moduli space of vector bundles over a family of curves (without formal trivialization data) satisfy a similar relation and, more relevantly, that the infinitesimal behavior of the latter is exactly Eq. (4.9).

Let  $\mathcal{M}_g$  denote the moduli space of genus  $g$  smooth projective curves over the field of complex numbers. Let  $\mathcal{M}_g^0$  denote the open subscheme in which a universal family exists (note that for  $g > 3$  it coincides with the moduli of curves without non-trivial automorphisms).

Let  $p_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{M}_g^0$  be the universal curve and let  $\mathcal{U}_{\mathcal{C}}(n, d) \rightarrow \mathcal{M}_g^0$  be the moduli space of rank- $n$  and degree- $d$  semistable vector bundles over  $\mathcal{C}$  [38, Theorem 1.21], [39, Application 2], [40, Section 2], [41, Theorem 5.3.2]. It is well known that in general the existence of a universal vector bundle on  $\mathcal{C} \times_{\mathcal{M}_g^0} \mathcal{U}_{\mathcal{C}}(n, d)$  may fail [42]. We are thus forced to consider the relative situation.

Let  $S$  be a scheme and let  $E$  be a relatively semistable vector bundle on  $\mathcal{C} \times_{\mathcal{M}_g^0} S$ . Recall that  $E$  yields a map  $S \rightarrow \mathcal{U}_{\mathcal{C}}(n, d)$ . Let  $\pi_{\mathcal{C}}, \pi$  be the projections of  $\mathcal{C} \times_{\mathcal{M}_g^0} S$  onto the first and second factors, respectively, and let  $p$  be the composition  $S \rightarrow \mathcal{U}_{\mathcal{C}}(n, d) \rightarrow \mathcal{M}_g^0$ .

**Theorem 5.1.** *For any integer  $\beta$ , consider the line bundle on  $S$  defined as follows*

$$\mathcal{L}_{\beta} := \text{Det } R^{\bullet} \pi_{*}(E \otimes \pi_{\mathcal{C}}^{*} \omega^{\otimes \beta}),$$

where  $\omega$  is the dualizing sheaf of  $\mathcal{C} \rightarrow \mathcal{M}_g^0$ .

Thus, there is an isomorphism of line bundles over  $S$ :

$$\mathcal{L}_{\beta} \xrightarrow{\sim} \mathcal{L}_1^{\otimes \beta} \otimes \mathcal{L}_0^{\otimes (1-\beta)} \otimes p^{*} \lambda_1^{\otimes 6n\beta(\beta-1)}, \quad (5.1)$$

where  $\lambda_1$  is the Hodge bundle on  $\mathcal{M}_g^0$ ; that is,  $\text{Det } R^{\bullet}(p_{\mathcal{C}})_{*} \omega$ .

**Proof.** Let  $K$  be an effective divisor associated with  $\omega$  and consider the following diagram:

$$\begin{array}{ccccc} & & \mathcal{C} \times_{\mathcal{M}_g^0} S & & \\ & \swarrow \pi_{\mathcal{C}} & & \searrow \pi & \\ K \subset \mathcal{C} & \xrightarrow{i} & \mathcal{C} & & S \\ & \searrow p_K & \swarrow p_{\mathcal{C}} & & \downarrow p \\ & & \mathcal{M}_g^0 & & \end{array}$$

where  $p_K$  is a finite and flat morphism of degree  $2g - 2$ . Observe that since  $p_K$  is finite there exists a line bundle  $N$  on  $\mathcal{M}_g^0$  such that  $i^{*} \omega = p_K^{*} N = i^{*} p_{\mathcal{C}}^{*} N$ , and therefore  $i^{*} \omega^{\otimes \beta} = i^{*} p_{\mathcal{C}}^{*} N^{\otimes \beta}$ . Observe that

$$i_{*} i^{*} \omega^{\otimes (\beta+1)} \xrightarrow{\sim} i_{*} (i^{*} p_{\mathcal{C}}^{*} N^{\otimes \beta} \otimes i^{*} \omega) \xrightarrow{\sim} i_{*} i^{*} (p_{\mathcal{C}}^{*} N^{\otimes \beta} \otimes \omega)$$

gives rise to an isomorphism between the cokernels of the following two exact sequences of bundles on  $\mathcal{C}$ :

$$\begin{aligned} 0 \rightarrow \omega^{\otimes \beta} \rightarrow \omega^{\otimes (\beta+1)} \rightarrow i_{*} i^{*} \omega^{\otimes (\beta+1)} \rightarrow 0 \\ 0 \rightarrow p_{\mathcal{C}}^{*} N^{\otimes \beta} \rightarrow p_{\mathcal{C}}^{*} N^{\otimes \beta} \otimes \omega \rightarrow i_{*} i^{*} (p_{\mathcal{C}}^{*} N^{\otimes \beta} \otimes \omega) \rightarrow 0. \end{aligned}$$

Taking the pullback by  $\pi_{\mathcal{C}}$ , tensoring by  $E$  and considering the determinant in these exact sequences, we obtain the isomorphism

$$\mathcal{L}_{\beta+1} \otimes \mathcal{L}_{\beta}^{-1} \xrightarrow{\sim} \text{Det } R^{\bullet} \pi_{*}(E \otimes \pi_{\mathcal{C}}^{*} \omega \otimes \pi_{\mathcal{C}}^{*} p_{\mathcal{C}}^{*} N^{\otimes \beta}) \otimes \text{Det } R^{\bullet} \pi_{*}(E \otimes \pi_{\mathcal{C}}^{*} p_{\mathcal{C}}^{*} N^{\otimes \beta})^{-1}.$$

Bearing in mind that  $\pi_{\mathcal{C}}^{*} p_{\mathcal{C}}^{*} N \simeq \pi^{*} p^{*} N$  and the properties of the determinant functor, the right-hand side is isomorphic to

$$\begin{aligned} \text{Det } R^{\bullet} \pi_{*}(E \otimes \pi_{\mathcal{C}}^{*} \omega) \otimes p_{\mathcal{C}}^{*} N^{\otimes \beta \cdot \chi(E \otimes \pi_{\mathcal{C}}^{*} \omega)} \otimes \text{Det } R^{\bullet} \pi_{*}(E)^{-1} \otimes (p_{\mathcal{C}}^{*} N^{\otimes \beta \cdot \chi(E)})^{-1} \xrightarrow{\sim} \\ \xrightarrow{\sim} \mathcal{L}_1 \otimes \mathcal{L}_0^{-1} \otimes p_{\mathcal{C}}^{*} N^{\otimes \beta n(2g-2)}, \end{aligned}$$

where  $\chi(M)$  is the Euler–Poincaré characteristic of the restriction of the sheaf  $M$  to the fibers of  $\mathcal{C} \times_{\mathcal{M}_g^0} S \rightarrow S$ . Summing up, we have proved that

$$\mathcal{L}_{\beta+1} \otimes \mathcal{L}_\beta^{-1} \xrightarrow{\sim} \mathcal{L}_1 \otimes \mathcal{L}_0^{-1} \otimes p_e^* N^{\otimes \beta n(2g-2)}. \quad (5.2)$$

Similar to the above, but replacing  $E$  by the trivial bundle, we obtain the following isomorphism:

$$p^* \lambda_{\beta+1} \otimes p^* \lambda_\beta^{-1} \xrightarrow{\sim} p^* \lambda_1 \otimes p^* \lambda_0^{-1} \otimes p_e^* N^{\otimes \beta n(2g-2)},$$

where  $\lambda_\beta := \text{Det } R^\bullet(p_e)_* \omega^{\otimes \beta}$ . Mumford's formula, which asserts that  $\lambda_\beta \simeq \lambda_1^{\otimes (6\beta^2 - 6\beta + 1)}$  [43], applied to the above formula yields the identity

$$p^* N^{\otimes \beta(2g-2)} \xrightarrow{\sim} p^* \lambda_1^{\otimes 12\beta}.$$

Substituting this into (5.2) yields

$$\mathcal{L}_{\beta+1} \xrightarrow{\sim} \mathcal{L}_\beta \otimes \mathcal{L}_1 \otimes \mathcal{L}_0^{-1} \otimes p^* \lambda_1^{\otimes 12n\beta}.$$

Proceeding recursively, we obtain the result

$$\begin{aligned} \mathcal{L}_{\beta+1} &\xrightarrow{\sim} \mathcal{L}_1^{\otimes \beta} \otimes (\mathcal{L}_0^{\otimes \beta})^{-1} \otimes \mathcal{L}_0 \otimes p^* \lambda_1^{\otimes \sum_{i=1}^{\beta} 12n(i-1)} \\ &\xrightarrow{\sim} \mathcal{L}_1^{\otimes \beta} \otimes \mathcal{L}_0^{\otimes (1-\beta)} \otimes p^* \lambda_1^{\otimes 6n\beta(\beta-1)}. \quad \square \end{aligned}$$

**Remark 5.2.** Schork proved that both sides of the isomorphism (5.1) have the same Chern class and offered an interpretation in terms of the  $bc$ -system of rank  $n$  [9].

The remainder of this section is devoted to showing that (4.9) and (5.1) are actually the same. To this end, we follow Muñoz Porras and Plaza Martín [8] for the case of the Mumford formula.

Let  $\mathcal{U}_g^\infty(n, d)_{ss}$  denote the subscheme of  $\mathcal{U}_g^\infty(n, d)$  consisting of those points for which the vector bundle is semistable. Note that there is a natural forgetful morphism

$$\mathcal{U}_g^\infty(n, d)_{ss} \longrightarrow \mathcal{U}_c(n, d)$$

that is a homomorphism of  $\mathcal{M}_g^0$ -schemes. If no confusion arises, we use a dashed arrow  $\mathcal{U}_g^\infty(n, d) \dashrightarrow \mathcal{U}_c(n, d)$  to denote the forgetful morphism defined only on the subscheme  $\mathcal{U}_g^\infty(n, d)_{ss}$ .

Continuing with the above notations, recall that  $S$  is a scheme and  $E$  is a relative semistable vector bundle on  $\mathcal{C} \times_{\mathcal{M}_g^0} S$  that yields a map  $S \rightarrow \mathcal{U}_c(n, d)$ . Assume that a lift of  $S \rightarrow \mathcal{U}_c(n, d)$  to  $\mathcal{U}_g^\infty(n, d)_{ss}$  is given; in other words, we have a five-tuple of the type  $(\mathcal{C} \times_{\mathcal{M}_g^0} S, x, z, E, \phi)$  or, equivalently, a commutative diagram

$$\begin{array}{ccc} & \mathcal{U}_g^\infty(n, d) & \\ \nearrow & \downarrow & \\ S & \longrightarrow & \mathcal{U}_c(n, d). \end{array}$$

Assume that a  $\mathbb{C}$ -valued point of  $S$  is given and let  $U \in \mathcal{U}_g^\infty(n, d)$  be the rational point of  $\text{Gr}(V)$  associated with it through the Krichever map. Using the results of Section 3, we have the following diagram

$$\begin{array}{ccccc} \text{SGL}_{\mathbb{C}((z))}(V) \simeq \text{SGL}_{\mathbb{C}((z))}(V) \times \{U\} & \xrightarrow{\mu_U} & \mathcal{U}_g^\infty(n, d) & & \\ \downarrow \mu_U & \nearrow & \downarrow & \searrow & \\ S & \xrightarrow{\bar{p}} & \mathcal{U}_c(n, d) & \longrightarrow & \mathcal{M}_g^0. \end{array}$$

We assume that the orbit of  $U$  under the action of  $\text{SGL}_{\mathbb{C}((z))}(V)$  falls inside  $S$ . Thus,  $\mu_U$  factors through  $S$  and we obtain  $\bar{\mu}_U$  (see the dashed arrow).

We now have the following result, which sheds light on the infinitesimal behavior of (5.1) at point  $U$ .

**Theorem 5.3.** *The pullback of (5.1)*

$$\mathcal{L}_\beta \xrightarrow{\sim} \mathcal{L}_1^{\otimes \beta} \otimes \mathcal{L}_0^{\otimes (1-\beta)} \otimes p^* \lambda_1^{\otimes 6n\beta(\beta-1)}$$

by  $\bar{\mu}_U$  is precisely (4.9):

$$\mathbb{L}_{n,\beta} \simeq \mathbb{L}_{n,1}^{\otimes \beta} \otimes \mathbb{L}_{n,0}^{\otimes (1-\beta)} \otimes p^* \Lambda_1^{\otimes 6n\beta(\beta-1)}.$$

**Proof.** With the above assumptions and notations, it suffices to prove that

$$\bar{\mu}_U^* \mathcal{L}_\beta \simeq \mathbb{L}_{n,\beta}$$

$$\bar{\mu}_U^* p^* \lambda_\beta \simeq \Lambda_\beta.$$

Recall from Section 4 the definition of the sheaves  $\mathbb{L}_{n,\beta}$  and  $\Lambda_n$ . Let  $\text{Kr}_{n,\beta}$  denote the modification of the Krichever map that sends  $(X, x, z, E, \phi)$  to  $H^0(X - \{x\}, E \otimes \omega_X^{\otimes \beta})$ . Thus, from the commutative diagram

$$\begin{array}{ccc} \text{SGl}_{\mathbb{C}((z))}(V) \simeq \text{SGl}_{\mathbb{C}((z))}(V) \times \{U\} & \xrightarrow{\mu_U} & \mathcal{U}_g^\infty(n, d) \xrightarrow{\text{Kr}_{n,\beta}} \text{Gr}(V) \\ \bar{\mu}_U \downarrow & \nearrow & \downarrow \\ S & \xrightarrow{\bar{p}} & \mathcal{U}_c(n, d) \end{array}$$

and the base-change property of the determinant, it is straightforward to see that

$$\mathbb{L}_{n,\beta} \simeq \mu_U^* \text{Kr}_{n,\beta}^* \text{Det} \simeq \bar{\mu}_U^* \mathcal{L}_\beta.$$

The second isomorphism follows analogously, which completes the proof.  $\square$

**Remark 5.4.** In cases in which a universal or Poincaré bundle exists on  $\mathcal{C} \times_{\mathcal{M}_g^0} \mathcal{U}_c(n, d)$  [42], we could repeat the above construction for  $S = \mathcal{U}_c(n, d)$  and  $E$  the universal object. Accordingly, Theorem 5.1 is indeed an identity that holds on the Picard group of  $\mathcal{U}_c(n, d)$  (see [44] for some facts on generators of this group).

## References

- [1] G. Segal, The definition of conformal field theory, 1990 (unpublished manuscript).
- [2] A. Pressley, G. Segal, Loop Groups, in: Oxford Mathematical Monographs, The Clarendon Press/ Oxford University Press, New York, 1986.
- [3] A. Beauville, Y. Laszlo, Conformal blocks and generalized theta functions, Comm. Math. Phys. 164 (1994) 385–419.
- [4] E. Frenkel, D. Ben-Zvi, Vertex Algebras and Algebraic Curves, second ed., in: Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, Providence, RI, 2004.
- [5] M. Mulase, Cohomological structure in soliton equations and Jacobian varieties, J. Differential Geom. 19 (1984) 403–430.
- [6] T. Shiota, Characterization of Jacobian varieties in terms of soliton equations, Invent. Math. 83 (1986) 333–382.
- [7] E. Arbarello, C. De Concini, V.G. Kac, C. Procesi, Moduli spaces of curves and representation theory, Comm. Math. Phys. 117 (1988) 1–36.
- [8] J.M. Muñoz Porras, F.J. Plaza Martín, Automorphism group of  $k((t))$ : applications to the bosonic string, Comm. Math. Phys. 216 (2001) 609–634.
- [9] M. Schork, A note on families of  $bc$ -systems of higher rank, J. Phys. A 34 (2001) L53–L57.
- [10] N. Kawamoto, Y. Namikawa, A. Tsuchiya, Y. Yamada, Geometric realization of conformal field theory on Riemann surfaces, Comm. Math. Phys. 116 (1988) 247–308.
- [11] P. Goddard, D. Olive, Kac–Moody and Virasoro algebras in relation to quantum physics, Internat. J. Modern Phys. A 1 (1986) 303–414.
- [12] A.A. Beĭlinson, V.V. Schechtman, Determinant bundles and Virasoro algebras, Comm. Math. Phys. 118 (1988) 651–701.
- [13] E. Frenkel, V. Kac, A. Radul, W. Wang,  $\mathcal{W}_{1+\infty}$  and  $\mathcal{W}(\mathfrak{gl}_N)$  with central charge  $N$ , Comm. Math. Phys. 170 (1995) 337–357.
- [14] V. Kac, A. Radul, Representation theory of the vertex algebra  $\mathcal{W}_{1+\infty}$ , Transform. Groups 1 (1996) 41–70.
- [15] H. Awata, M. Fukuma, Y. Matsuo, S. Odake, Representation theory of the  $\mathcal{W}_{1+\infty}$  algebra, Progr. Theoret. Phys. Suppl. 118 (1995) 343–373.
- [16] M. Sato, Y. Sato, Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold, North-Holland Math. Stud. 81 (1983) 259–271.
- [17] G. Segal, G. Wilson, Loop groups and equations of KdV type, Publ. Math. Inst. Hautes Études Sci. 61 (1985) 5–65.
- [18] M. Mulase, Category of vector bundles on algebraic curves and infinite-dimensional Grassmannians, Int. J. Math. 1 (1990) 293–342.
- [19] J.M. Muñoz Porras, F.J. Plaza Martín, Equations of the moduli of pointed curves in the infinite Grassmannian, J. Differential Geom. 51 (1999) 431–469.
- [20] A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV, Publ. Math. Inst. Hautes Études Sci. 32 (1967) 5–361.
- [21] C.S. Seshadri, Fibrés vectoriels sur les courbes algébriques, in: Astérisque, vol. 96, Société Mathématique de France, Paris, 1982.
- [22] I. Biswas, G. Schumacher, Determinant bundle, Quillen metric, and Peterson–Weil form on moduli spaces, Geom. Funct. Anal. 9 (1999) 226–255.
- [23] G.E. Welters, Polarized Abelian varieties and the heat equations, Compos. Math. 49 (1983) 173–194.
- [24] S. Li, On the deformation theory of pair  $(x, e)$ , Preprint [math.AG/0809.0344v1](https://arxiv.org/abs/math/0809.0344v1).
- [25] J.M. Muñoz Porras, F.J. Plaza Martín, Coverings with prescribed ramification and Virasoro groups, Kyoto J. Math. 50 (2010) 51–73.
- [26] K.S. Brown, Cohomology of Groups, in: Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994.
- [27] V.G. Kac, A.K. Raina, Bombay Lectures on Highest Weight Representations of Infinite-dimensional Lie Algebras, in: Advanced Series in Mathematical Physics, vol. 2, World Scientific Publishing, Teaneck, NJ, 1987.
- [28] G. Falqui, C. Reina, Conformal field theory and moduli spaces of vector bundles over variable Riemann surfaces, Lecture Notes in Phys. 375 (1991) 209–218.
- [29] I. Biswas, A.K. Raina, Projective structures on a Riemann surface, Int. Math. Res. Not. 15 (1996) 753–768.
- [30] A.M. Semikhatov, Solving Virasoro constraints on integrable hierarchies via the Kontsevich–Miwa transform, Nuclear Phys. B 386 (1992) 139–165.
- [31] A. Grothendieck, Groupes de monodromie en géométrie algébrique. I. Lecture Notes in Math. 288 (1972) 1–24.
- [32] W. Li, R.L. Wilson, Central extensions of some Lie algebras, Proc. Amer. Math. Soc. 126 (1998) 2569–2577.
- [33] A.O. Radul, Central extension of Lie algebra of differential operators on a circle and  $W$  algebras, Pis'ma Zh. Èksp. Teor. Fiz. 50 (1989) 341–343.
- [34] V.G. Kac, D.H. Peterson, Spin and wedge representations of infinite-dimensional Lie algebras and groups, Proc. Natl. Acad. Sci. USA 78 (1981) 3308–3312.
- [35] L.A. Dickey, Lectures on classical  $W$ -algebras, Acta Appl. Math. 47 (1997) 243–321.
- [36] G. Compère, S. Detournay, Semi-classical central charge in topologically massive gravity, Class. Quantum Gravity 26 (2009) 012001.

- [37] M. Schlichenmaier, Local cocycles and central extensions for multipoint algebras of Krichever–Novikov type, *J. Reine Angew. Math.* 559 (2003) 53–94.
- [38] C.T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety. I, *Publ. Math. Inst. Hautes Études Sci.* 79 (1994) 47–129.
- [39] S. Mukai, Symplectic structure of the moduli space of sheaves on an Abelian or  $K3$  surface, *Invent. Math.* 77 (1984) 101–116.
- [40] T. Bridgeland, A. Maciocia, Fourier–Mukai transforms for  $K3$  and elliptic fibrations, *J. Algebraic. Geom.* 11 (2002) 629–657.
- [41] T. Bridgeland, Fourier–Mukai transforms for surfaces and moduli spaces of stable sheaves, Ph.D. Thesis, 1998.
- [42] N. Mestrano, S. Ramanan, Poincaré bundles for families of curves, *J. Reine Angew. Math.* 362 (1985) 169–178.
- [43] D. Mumford, Stability of projective varieties, *Enseign. Math.* 23 (2) (1977) 39–110.
- [44] A. Kouvidakis, On the moduli space of vector bundles on the fibers of the universal curve, *J. Differential Geom.* 37 (1993) 505–522.