

III Iberoamerican Congress on Geometry
Salamanca, June 8th-12th, 2004

A solution of the
Schottky-Type problem
for curves with automorphisms

joint work with:

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Schottky problem:

- Give conditions that a theta function of a principally polarized abelian variety (p.p.a.v.) should satisfy in order to be a jacobian theta function of a Riemann surface.

Theorem. (Shiota)

Let X be a irreducible p.p.a.v. of dimension g . Then, the following conditions are equivalent

- 1. X is isomorphic to the Jacobian variety of a complete smooth curve of genus g .*
- 2. there exists $a_1, a_2, a_3 \in \mathbb{C}^g$ and a quadratic form $Q(t)$ such that for any $\xi \in \mathbb{C}^g$ the function*

$$\tau(t) := \exp(Q(t))\theta(t_1a_1 + t_2a_2 + t_3a_3 + \xi)$$

solves the KP equation.

- 3. there exists a $g \times \infty$ -matrix A of rank g and a quadratic form $Q(t)$ such that for any $\xi \in \mathbb{C}^g$ the function*

$$\tau(t) := \exp(Q(t))\theta(At + \xi)$$

is a τ -function for the KP hierarchy.

Schottky problem for R.S. with automorphisms:

- Give conditions that a theta function of a p.p.a.v. should satisfy in order to be a jacobian theta function of a Riemann surface with a non-trivial automorphism.

Theorem.

Let X be an irreducible p.p.a.v. of dimension g . Then the following conditions are equivalent.

1. *There exists a projective irreducible smooth genus g curve C with a non-trivial automorphism $\sigma_C : C \rightarrow C$ such that X is isomorphic as a p.p.a.v. to the Jacobian of C .*
2. *There exist a prime number p , p matrices $A^{(1)}, \dots, A^{(p)}$ ($A^{(j)}$ being a $g \times \infty$ -matrix of rank g) and p symmetric quadratic forms $Q^{(1)}, \dots, Q^{(p)}$, such that*
 - a) *for a $\xi_0 \in \mathbb{C}^g$, the corresponding BA-functions satisfy the $(1, \dots, 1)$ -KP hierarchy*

$$\text{Res} \left(\sum_{j=1}^p z^{-\delta_{ju}-\delta_{jv}} \psi_{u,\xi_0}^{(j)}(z, t) \psi_{v,\xi_0}^{*(j)}(z, s) \right) dz = 0$$

- b) *there exist $\xi_1 \in \mathbb{C}^g$ (depending on ξ_0) such that*

$$\text{Res} \left(\sum_{j=1}^p z^{-\delta_{ju}-\delta_{jv}} \psi_{v+1,\xi_0}^{(j+1)}(z, \sigma^*(t)) \psi_{u,\xi_1}^{*(j)}(z, s) \right) dz = 0$$

where $\sigma^*(t) := (t^{(p)}, t^{(1)}, t^{(2)}, \dots, t^{(p-1)})$.

PLAN OF THE TALK

1. Preliminaries on Grassmannians, formal groups, τ -functions, BA-functions and bilinear identities.
2. Geometric meaning of Γ_V and τ -function in the Jacobian case. . . .
3. Characterization of Jacobians of curves with automorphisms as finite dimensional orbits.
4. Main Theorem.

1. PRELIMINARIES

$$\begin{aligned}
V &= \mathbf{C}((z_1)) \times \cdots \times \mathbf{C}((z_p)) \\
z &= (z_1, \dots, z_p) \\
V^+ &= \mathbf{C}[[z_1]] \times \cdots \times \mathbf{C}[[z_p]] \\
V^- &= z_1^{-1}\mathbf{C}[z_1^{-1}] \times \cdots \times z_p^{-1}\mathbf{C}[z_p^{-1}] \\
\sigma(z_i) &= z_{i+1} \quad \sigma(z_p) = z_1
\end{aligned}$$

Consider the formal group scheme (R a $\mathbf{C}$ -algebra)

$$\Gamma_V(R) := \text{conn. comp. of } 1 \text{ in } (V \hat{\otimes}_{\mathbf{C}} R)^*$$

Then, $\Gamma_V = \Gamma_V^- \times \bar{\Gamma}_V^+$ where

$$\begin{aligned}
\bar{\Gamma}_V^+(R) &:= (V^+ \hat{\otimes}_{\mathbf{C}} R)^* \\
\Gamma_V^-(R) &:= ((1 + V^-) \hat{\otimes}_{\mathbf{C}} R)^*
\end{aligned}$$

The Abel morphism

$$\hat{C}_V := \mathbf{C}[[\bar{z}_1]] \times \cdots \times \mathbf{C}[[\bar{z}_p]] \xrightarrow{\phi_V} \Gamma_V^-$$

is the corresponding to the r -tuple

$$\left(\left(1 - \frac{\bar{z}_1}{z_1} \right)^{-1}, \dots, \left(1 - \frac{\bar{z}_r}{z_r} \right)^{-1} \right)$$

Then, (Γ_V^-, ϕ_V) is the Albanese variety of $\hat{C}_V$.

The infinite Grassmannian scheme

$$\mathrm{Gr}(V) := \left\{ U \subset V \text{ such that } U \rightarrow V/V^+ \text{ has} \right. \\ \left. \text{finite dimensional ker and coker} \right\}$$

endowed with the determinant bundle

$$\mathrm{Det}_V = \mathrm{Det}(\mathcal{L} \longrightarrow V/V^+)$$

and a canonical section

$$\Omega_+ \in H^0(\mathrm{Gr}^0(V), \mathrm{Det}_V^*) .$$

The group Γ_V acts on $\mathrm{Gr}(V)$ preserving Det_V .

The τ -function of $U \in \mathrm{Gr}(V)$ is

$$\tau_U(g) = \frac{\Omega_+(gU)}{g\delta_U} \quad g \in \Gamma_V^-$$

The u -th Baker-Akhiezer function of a point $U \in \mathrm{Gr}(V)$ is the V -valued function $\psi_{u,U}(z., t)$

$$\left(\exp \left(- \sum_{i \geq 1} \frac{t_i^{(1)}}{z_1^i} \right) \frac{\tau_{U_{u1}}(t + [z_1])}{\tau_U(t)}, \dots, \exp \left(- \sum_{i \geq 1} \frac{t_i^{(r)}}{z_r^i} \right) \frac{\tau_{U_{ur}}(t + [z_r])}{\tau_U(t)} \right)$$

Consider the following pairing

$$\begin{aligned} V \times V &\longrightarrow \mathbf{C} \\ (a, b) &\longmapsto \operatorname{Res}_{z=0} \operatorname{Tr}(a, b) dz . \end{aligned}$$

Define the adjoint Baker-Akhiezer function

$$\psi_{u,U}^*(z, t) := \psi_{u,U^\perp}(z, -t)$$

Then, one obtains that the the BA functions satisfy the $(1, ., 1)$ -KP-hierarchy

$$\boxed{\operatorname{Res}_{z=0} \operatorname{Tr} \left(\frac{\psi_{u,U}(z, t)}{(1, \dots, z_u, \dots, 1)} \cdot \frac{\psi_{v,U}^*(z, s)}{(1, \dots, z_v, \dots, 1)} \right) dz = 0}$$

for all $u, v \in \{1, \dots, p\}$. Equivalently, in terms of p.d.e.'s for the τ -function.

$$\begin{aligned} \sum_{j=1}^p \sum_{\substack{\beta_1 + \beta_2 - \alpha_1 - \alpha_2 \\ = \delta_{ju} + \delta_{jv} - 1}} p_{\alpha_1}(-t^{(j)}) p_{\beta_1}(\tilde{\partial}_{t^{(j)}}) \tau_{U_{u,j}}(t) \\ \cdot p_{\alpha_2}(s^{(j)}) p_{\beta_2}(-\tilde{\partial}_{s^{(j)}}) \tau_{U_{jv}}(s) = 0 \end{aligned}$$

2. Geometric interpretation

Fix data $(C, \bar{x}, t_{\bar{x}})$ where

C a proper smooth irreducible curve

$\bar{x} = \{x_1, \dots, x_r\}$ pairwise distinct smooth points

$t_{\bar{x}} = \{t_{x_1}, \dots, t_{x_r}\}$ isomorphisms $t_{x_j} : \hat{\mathcal{O}}_{C, x_j} \xrightarrow{\sim} \mathbb{C}[[z_j]]$

Given $L \in J_{g-1}(C)$, there is a canonical commutative diagram:

$$\begin{array}{ccc} \hat{C}_V & \xrightarrow{\alpha} & C \\ \phi_V \downarrow & & \downarrow \phi_L \\ \Gamma_V^- & \xrightarrow{\hat{\gamma}} & J_{g-1}(C) \end{array}$$

and a canonical diagram

$$\begin{array}{ccc} \Gamma_V & \xrightarrow{\hat{\gamma}} & J_{g-1}^\infty(C, \bar{x}) \xrightarrow{\text{Kr}} \text{Gr}(V) \\ & & \downarrow \pi \\ & & J_{g-1}(C) \end{array}$$

where $J_{g-1}^\infty(C, \bar{x})$ parametrizes pairs (L, ϕ) (with $\phi : \hat{L}_{\bar{x}} \xrightarrow{\sim} \hat{\mathcal{O}}_{C, \bar{x}}$) and Kr is the Krichever morphism

$$\text{Kr}(L, \phi) := (t_{\bar{x}} \circ \phi)(H^0(C - \bar{x}, L)) \subset V$$

It holds

$$\text{Kr}^* \text{Det}_V^* \simeq \pi^* \mathcal{O}_{J_{g-1}}(\Theta)$$

and therefore, a relation between the τ -function and the theta function of $J_{g-1}(C)$.

$\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ symplectic basis for $H_1(C, \mathbf{Z})$

$\{\omega_1, \dots, \omega_g\}$ holomorphic 1-forms with $\int_{\alpha_i} \omega_j = \delta_{ij}$

Ω the period matrix; that is, $\Omega_{ij} = \int_{\beta_i} \omega_j$

$$\theta(z) = \sum_{m \in \mathbf{Z}^g} \exp(2\pi i m^t z + \pi i m^t \Omega m)$$

$A^{(j)}: H^0(C, \Omega_C) \rightarrow \mathbf{C}[[z_j]]dz_j$ maps ω_i to $(\sum_k a_{ik}^{(j)})dz_j$

For $j \in \{1, \dots, r\}$ and $n > 0$, let $\eta_n^{(j)}$ be the unique meromorphic 1-form such that

$$\int^x \eta_n^{(j)} = z_j^{-n} - 2 \sum_{m=1}^{\infty} q_{nm}^{(j)} \frac{z_j^m}{m} \quad \text{at } x = x_j .$$

Define the τ -function as

$$\tau(\xi, t) := \exp(Q(t)) \theta(A(t) + \xi)$$

where

$$t = (t^{(1)}, \dots, t^{(r)}) \text{ and } t^{(j)} = (t_1^{(j)}, t_2^{(j)}, \dots)$$

$$Q(t) = \sum_{n,m \geq 1} q_{nm}^{(1)} t_n^{(1)} t_m^{(1)} + \dots + \sum_{n,m \geq 1} q_{nm}^{(r)} t_n^{(r)} t_m^{(r)}$$

$$A(t) = A^{(1)} t^{(1)} + \dots + A^{(r)} t^{(r)}$$

From the τ -function, BA functions are introduced and they satisfy the $(1, \dots, 1)$ -KP-hierarchy

$$\text{Res}_{z=0} \text{Tr} \left(\frac{\psi_{u,U}(z, t)}{(1, \dots, z_u, \dots, 1)} \cdot \frac{\psi_{v,U}^*(z, s)}{(1, \dots, z_v, \dots, 1)} \right) dz = 0$$

3. Finite dimensional orbits

Consider the functor whose rational points are

$$\{(C, \sigma_C, \bar{x}, t_{\bar{x}}, L, \phi_{\bar{x}})\} / \text{equiv}$$

where

$(C, \bar{x}, t_{\bar{x}})$ as above,

σ_C an order p automorphism of C , $\sigma_C(x_i) = x_{i+1}$,

$(L, \phi_{\bar{x}})$ a line bundle with a triv. along $\bar{x}$

Theorem. *Via the Krichever map*

$$(C, \sigma_C, \bar{x}, t_{\bar{x}}, L, \phi_{\bar{x}}) \mapsto (t_{\bar{x}} \circ \phi_{\bar{x}})H^0(C - \bar{x}, L) \subset V$$

the previous functor is representable by a subscheme $\text{Pic}^\infty(p)$ of $\text{Gr}(V)$.

Γ_V acts on $\text{Gr}(V)^p := \text{Gr}(V) \times \dots \times \text{Gr}(V)$

$$\mu^p: \Gamma_V \times \text{Gr}(V)^p \longrightarrow \text{Gr}(V)^p$$

$$(g, (U_1, \dots, U_p)) \mapsto (gU_1, \dots, gU_p) .$$

One has a Γ_V -equivariant closed immersion

$$\begin{aligned} \mathrm{Gr}(V) &\hookrightarrow \mathrm{Gr}(V)^p \\ U &\mapsto U_\sigma := (U, \sigma(U), \dots, \sigma^{p-1}(U)) \end{aligned}$$

$\Gamma_V(U_\sigma)$ is the orbit of $U_\sigma \in \mathrm{Gr}(V)^p$ under Γ_V .

$\Gamma_V(U_\sigma)/\bar{\Gamma}_V^+$ is a formal scheme.

Theorem. *Let U be a closed point of $\mathrm{Gr}(V)$. Then the following conditions are equivalent.*

1. $\dim_{\mathbb{C}} T_{U_\sigma}(\Gamma_V(U_\sigma)/\bar{\Gamma}_V^+) < \infty$,
2. *there exists $(C, \sigma_C, \bar{x}, t_x, L, \phi_x) \in \mathrm{Pic}^\infty(p)$ such that its image by the Krichever morphism is U .*

Proof. *It holds*

$$T_{U_\sigma}(\Gamma_V(U_\sigma)/\bar{\Gamma}_V^+) \simeq T_1\Gamma_V/(\ker d\mu_{U_\sigma}^p + T_1\bar{\Gamma}_V^+),$$

where $d\mu_{U_\sigma}^p: T_1\Gamma_V \longrightarrow T_{U_\sigma}\mathrm{Gr}(V)^p$.

$$\ker d\mu_{U_\sigma}^p = \cap_{i=0}^{p-1} \sigma^i A_U$$

$$A_U := \{g \in \Gamma_V \mid g \cdot U \subseteq U\}$$

4. Main Theorem

Let $X_\Omega := \mathbf{C}^g / (\mathbf{Z}^g + \Omega \mathbf{Z}^g)$ an irreducible p.p.a.v. And let $\theta(z) = \theta(z, \Omega)$ denote its Riemann theta function.

Let r be a natural number. Let $A^{(j)}$ be a $g \times \infty$ -matrix of rank g for each $j \in \{1, \dots, r\}$. Let $Q^{(j)}$ be a quadratic form for each $j \in \{1, \dots, r\}$. Let ξ be in $\mathbf{C}^g$.

Given $(X, \Omega, \{A^{(1)}, \dots, A^{(r)}\}, \{Q^{(1)}, \dots, Q^{(r)}\}, \xi)$, its τ -function, $\tau(\xi, t)$, and BA-functions, $\psi_{u, \xi}(z, t)$, are given formally (analogously as in the Jacobian case).

Theorem. *Then the following conditions are equivalent.*

1. *There exists $(C, \bar{x}, t_{\bar{x}})$ s.t. $X_\Omega \simeq J(C)$.*
2. *There exist a $A^{(j)}$ and $Q^{(j)}$ s.t. $\tau(\xi, t)$ is a τ -function of the $(1, \dots, 1)$ -KP-hierarchy for all $\xi \in \mathbf{C}^g$.*

1) $\Rightarrow$ 2) It follows from the section 4.

2) $\Rightarrow$ 1) We denote

$$A(t) = \sum_{j=1}^r A^{(j)} t^{(j)} \quad \text{and} \quad Q(t) = \sum_{j=1}^r \sum_{i, k \geq 1} q_{ik}^{(j)} t_i^{(j)} t_k^{(j)} .$$

Since $\tau(\xi, t)$ is a τ -function of the $(1, \dots, 1)$ -KP-hierarchy, it defines a point $U_\xi \in \text{Gr}(V)$ s.t. $\tau(\xi, t) = \tau_{U_\xi}(t)$. Therefore, we have obtained a morphism

$$\begin{aligned} \varphi: \mathbf{C}^g &\longrightarrow \text{Gr}(V) \\ \xi &\longmapsto U_\xi . \end{aligned}$$

Since Θ is a principal polarization on X , one gets an injection

$$X \hookrightarrow \text{Gr}(V) . \quad (1)$$

The surjective map

$$T_1 \Gamma_V^- = \mathbf{C}^\infty \times \dots \times \mathbf{C}^\infty \xrightarrow{A} \mathbf{C}^g = T_\xi X$$

yields

$$\Gamma_V^- \xrightarrow{A_\xi} \widehat{X}_\xi .$$

The properties of τ -functions imply that there is a factorization

$$\begin{array}{ccc} \Gamma_V^- & \xrightarrow{\mu_\xi} & \Gamma(U_\xi) / \bar{\Gamma}_V^+ \\ & \searrow A_\xi & \nearrow \\ & \widehat{X}_\xi & \end{array}$$

It follows that $\dim T_{U_\xi} \Gamma(U_\xi) / \bar{\Gamma}_V^+$ is finite.

By Mulase, $\exists (C_\xi, \bar{x}_\xi, t_\xi, L_\xi, \phi_\xi)$ associated to U_ξ under the Krichever map. Furthermore, $(C_\xi, \bar{x}_\xi, t_\xi)$ does not depend on ξ .

Then, the fact that $\text{rank } A^{(j)} = g$ says that the image of

$$\begin{array}{ccc} X & \hookrightarrow & \overline{\text{Pic}}(C) \\ \xi' & \longmapsto & L_{\xi'} \end{array}$$

is $\text{Pic}^0(C) \cdot L_{\xi}$.

So $\text{Pic}^0(C)$ is a quotient of an abelian variety $\implies C$ is a smooth complete curve of genus at most g .

To finish the implication $2) \Rightarrow 1)$, all that is left to show that $X \rightarrow \text{Pic}^0(C)$ is an isomorphism of p.p.a.v.'s, which we now do.

Let $\tau_X = \tau(\xi, t)$ be the τ -function of $X, \dots$

Let $\tau_J = \tau(\xi, t)$ be the τ -function of $J(C), \dots$

From the construction $\tau_X = \tau_J$ and hence

$$\theta_X(A(t) + \xi) = \exp(q(t)) \theta_J(A_J(t) + \xi)$$

where $q(t)$ is a quadratic function.

Therefore

$$\Theta_X = \Theta_J$$

In particular, the curve C is irreducible and of genus g .

5. Szegő Kernel