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Infinite Grassmannians and Applications to Arithmetic

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Aim.

Weil Reciprocity Law [Weil, Serre, etc.]

Let X be a smooth, complete and connected curve over an algebraically closed field k . Given two rational functions f and g in X with disjoint supports, it holds that

$$\prod_{x \in X} f(x)^{v_x(g)} = \prod_{x \in X} g(x)^{v_x(f)}$$

And, more generally

$$1 = \prod_{\substack{x \in X \\ x \text{ closed}}} \text{Norm}_{k(x)/k} \left((-1)^{v_x(f)v_x(g)} \left(\frac{f^{v_x(g)}}{g^{v_x(f)}} \right)(x) \right)$$

Following ideas of Tate and Arbarello-De Concini-Kac, we aim at studying this expression and obtain new results by using the techniques of Infinite Grassmannians (Sato-Sato and Segal-Wilson).

Applications in progress

We will also discuss the case of perfect fields as well as some applications to the study of the group $\mathbb{I}_X / \Sigma_X^*$ (idele group mod group of invertible functions).

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Infinite Grassmannians

Infinite grassmannians appeared in the decade of 1980's (Segal-Wilson and Sato-Sato). They arose as a powerful tool when studying:

- commuting rings of differential operators (Burchnell-Chaundy, Mumford);
- KP hierarchy (Sato-Sato, Mulase).
- Schottky problem (Shiota);

More explicitly. The pair $(V = k((z)), V^+ = k[[z]])$ defines a linear topology in V by considering as neighborhoods of 0 those subspaces $A \subset V$ s.t. $(A + V^+)/ (A \cap V^+)$ is finite dimensional. We observe that V is Hausdorff and complete w.r.t. this topology.

Theorem [Álvarez-Muñoz-P.]

There is a k -scheme $\text{Gr}(V, V^+)$, the infinite grassmannian, whose set of k -rational points is:

$$\text{Gr}(V, V^+) := \left\{ \begin{array}{l} U \subset V \text{ s.t. } U \rightarrow V/V^+ \\ \text{has finite dimensional ker and coker} \end{array} \right\}$$

Krichever Map

Let us fix (X, x, t) . Then, for each pair (L, ϕ) , where:

- L is a line bundle of degree $g - 1$,
- $\phi : \widehat{L}_x \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$,

the cohomology exact sequence:

$$0 \rightarrow H^0(X, L) \rightarrow H^0(X - x, L) \rightarrow V/V^+ \rightarrow H^1(X, L) \rightarrow 0$$

where $V := (\widehat{\mathcal{O}}_{X,x})_{(0)}$, $V^+ := \widehat{\mathcal{O}}_{X,x}$, shows that:

$$(L, \phi) \mapsto (t \circ \phi)(H^0(X - x, L)) \in \text{Gr}(V)$$

is well defined and it will be called the **Krichever Map**.

Theorem [Álvarez-Muñoz-P.]

$\text{Jac}^\infty(X) := \{(L, \phi) \text{ line bundle and triv.}\}$ is representable by a closed subscheme of $\text{Gr}(V)$.

One could also vary (X, x, t) or consider higher rank vector bundles.

Examples

EXAMPLE 1. Let $X = \mathbb{P}^1 = \text{Proj } \mathbb{C}[x_0, x_1]$, $x = \infty = (0, 1)$ be the point given by the equation $x_0 = 0$ and let t be the isomorphism $\hat{\mathcal{O}}_{\mathbb{P}^1, \infty} \simeq \mathbb{C}[[z]]$ given by $\frac{x_0}{x_1} \mapsto z$.

Then,

$$H^0(\mathbb{P}^1 \setminus \{\infty\}, \mathcal{O}_{\mathbb{P}^1}) \simeq \mathbb{C}\left[\frac{x_1}{x_0}\right] \mapsto \mathbb{C}[z^{-1}] \in \text{Gr}^1(\mathbb{C}((z)))$$

$(\mathbb{P}^1, \infty, t)$ can be reconstructed from its image by the Krichever map, $\mathbb{C}[z^{-1}]$. Namely, $\mathbb{P}^1 \setminus \{\infty\} \simeq \text{Spec } \mathbb{C}[z^{-1}]$, ...

EXAMPLE 2. With the above notation, consider the point $(1, 0) \in \mathbb{P}^1$, and the line bundle $\mathcal{O}_{\mathbb{P}^1}(2 \cdot (1, 0))$, whose Euler-Poincaré characteristic is 3, then:

$$H^0(\mathbb{P}^1 \setminus \{\infty\}, \mathcal{O}_{\mathbb{P}^1}(2 \cdot (1, 0))) \simeq \left(\frac{x_0}{x_1}\right)^2 \mathbb{C}\left[\frac{x_1}{x_0}\right] \mapsto z^2 \mathbb{C}[z^{-1}] \in \text{Gr}^3(\mathbb{C}((z)))$$

(the trivialization ϕ corresponds to multiplication by $(\frac{x_1}{x_0})^2$).

It is trivial that $\mathcal{O}_{\mathbb{P}^1}(2 \cdot (1, 0))$ and ϕ can be reconstructed from $z^2 \mathbb{C}[z^{-1}]$ as $\mathbb{C}[z^{-1}]$ -module.



Determinant

Similarly to the finite dimensional grassmannians, $\text{Gr}(V)$ carries a determinant bundle as well as an action of the general linear group. The determinant line bundle, Det , is the determinant of the perfect complex of $\mathcal{O}_{\text{Gr}(V)}$ -modules:

$$\mathcal{U} \oplus V^+ \longrightarrow V$$

where $\mathcal{U}$ is the universal object.

REMARK The fiber of the determinant line bundle, Det , at a point U is:

$$\text{Det}_U = \wedge^{\text{top}}(U \cap V^+) \otimes \wedge^{\text{top}}(V/(U + V^+))^*$$

In particular, if U is the image of (L, ϕ) under the Krichever map and $\deg(L) = g - 1$, then:

$$\text{Det}_U \simeq \wedge^{\text{top}} H^0(X, L) \otimes \wedge^{\text{top}} H^1(X, L)^*$$



Central Extensions

Define the linear group by:

$$\mathrm{Gl}(V) := \{g : V \xrightarrow{\sim} V \text{ s.t. } g, g^{-1} \text{ are continuous} \}$$

The action of $\mathrm{Gl}(V)$ on $\mathrm{Gr}(V)$ preserves the determinant bundle:

$$g^* \mathrm{Det} \simeq \mathrm{Det} \quad \text{for all } g \in \mathrm{Gl}(V)$$

It follows, since $H^0(\mathrm{Gr}(V), \mathcal{O}^*) = k^*$, that for any group $G \subseteq \mathrm{Gl}(V)$ one may consider the **central extension**:

$$1 \rightarrow k^* \rightarrow \tilde{G} := \left\{ \begin{array}{ccc} \mathrm{Det} & \xrightarrow[\sim]{\tilde{g}} & \mathrm{Det} \\ \downarrow & & \downarrow \\ \mathrm{Gr}(V) & \xrightarrow[\sim]{g} & \mathrm{Gr}(V) \end{array} \text{ with } g \in G \right\} \xrightarrow{\pi} G \rightarrow 1$$

This extension is characterized by its cocycle, whose explicit expression is $c(g_1, g_2) = \det(\tilde{g}_1 \circ (\tilde{g}_1 \circ \tilde{g}_2)^{-1} \circ \tilde{g}_2)$ for $(\tilde{g}_i, g_i) \in \tilde{G}$.

These ideas, dating back to Tate, provide an alternative approach to the study of **reciprocity laws** [Anderson-Pablos, Muñoz-Pablos, etc.].

Weil Reciprocity Law

Weil Reciprocity Law

Let X be a smooth, complete and connected curve and k be algebraically closed field. Given rational functions f and g in X with disjoint supports, it holds that:

$$\prod_{x \in X} f(x)^{v_x(g)} = \prod_{x \in X} g(x)^{v_x(f)}$$

EXAMPLE. $k = \mathbb{C}$, $X = \mathbb{P}_1$, $\alpha_i, \beta_i, \gamma_j, \delta_j$ distinct, and

$$f(z) = a \prod_{i=1}^m \frac{z - \alpha_i}{z - \beta_i}, \quad g(z) = b \prod_{j=1}^n \frac{z - \gamma_j}{z - \delta_j}$$

It holds that $v_x(g) = 1$ for $x = \gamma_j$, $v_x(g) = -1$ for $x = \delta_j$, and $v_x(g) = 0$ otherwise; that is, $\prod_{x \in X} f(x)^{v_x(g)} = \prod_{j=1}^m f(\gamma_j) f(\delta_j)^{-1}$. The law reads

$$\prod_{j=1}^n \prod_{i=1}^m \frac{\gamma_j - \alpha_i}{\gamma_j - \beta_i} \left(\frac{\delta_j - \alpha_i}{\delta_j - \beta_i} \right)^{-1} = \prod_{i=1}^m \prod_{j=1}^n \frac{\alpha_i - \gamma_j}{\alpha_i - \delta_j} \left(\frac{\beta_i - \gamma_j}{\beta_i - \delta_j} \right)^{-1}$$

There are generalizations for k non algebraically closed and f, g arbitrary.



Adeles

Notation

- k a perfect field.
- X a smooth, complete and connected curve over k .
- Σ_X the function field of X .
- for a closed point $x \in X$, $k(x)$ is its residue field, $A_x = \widehat{\mathcal{O}_{X,x}}$ is the completion of the local ring, $\mathfrak{m}_x$ its maximal ideal and $K_x = (A_x)_{(0)}$.
- $\mathbb{A}_X := \prod'_{x \in X} K_x = \{(\alpha_x)_{x \in X} \text{ s.t. } \alpha_x \in A_x \text{ for almost all } x\}$
- $\mathbb{A}_X^+ := \prod_{x \in X} A_x$,
- $\mathbb{I}_X$ the idele group (the group of invertible elements of $\mathbb{A}_X$).

Hence, we may consider infinite grassmannians corresponding to different choices of vector spaces, namely:

- local case: $\text{Gr}(K_x, A_x)$ for $x \in X$;
- global/adelic case: $\text{Gr}(\mathbb{A}_X, \mathbb{A}_X^+)$.

The relations between the associated determinant bundles and commutators (of central extensions) yield an abstract reciprocity law for the tame symbol ([Muñoz-Pablos]).

Local Case

Tame Symbol

Tame Symbol

For $x \in X$ with residue field $k(x)$, the tame symbol of f, g is:

$$\Sigma_X^* \times \Sigma_X^* \longrightarrow k(x)^*$$

$$(f, g) \longmapsto (f, g)_x := \text{class of } (-1)^{v_x(f)v_x(g)} \frac{f^{v_x(g)}}{g^{v_x(f)}}$$

$\Sigma_X^* \hookrightarrow K_x^*$ and acts on $\text{Gr}(K_x, A_x)$ by homotheties. Hence, one has:

$$1 \rightarrow k^* \rightarrow \widetilde{\Sigma_X^*} \xrightarrow{\pi_x} \Sigma_X^* \rightarrow 1$$

and, since $f, g \in \Sigma_X^*$ commute, the commutator makes sense and is:

$$\{f, g\}_{A_x}^{K_x} := \tilde{f} \cdot \tilde{g} \cdot \tilde{f}^{-1} \cdot \tilde{g}^{-1} \in k^*$$

Let us compute explicitly the commutator.



Local Case

Commutator

Given $f, g \in \Sigma_X$, consider $\tilde{f}, \tilde{g} \in \tilde{\Sigma}_X^*$ with $\pi(\tilde{f}) = f, \pi(\tilde{g}) = g$.

Observe that $\tilde{f} : f^* \text{Det} \xrightarrow{\sim} \text{Det}$ and that

$$\begin{aligned} (f^* \text{Det})_U &\simeq \text{Det}_{f(U)} \simeq \wedge^{\text{top}}(U \cap f^{-1}V^+) \otimes \wedge^{\text{top}}(V/(U + f^{-1}V^+))^* \simeq \\ &\simeq \wedge^{\text{top}}(U \cap V^+) \otimes \wedge^{\text{top}}(V/(U + V^+))^* \otimes \\ &\quad \otimes \wedge^{\text{top}}(f^{-1}V^+/(V^+ \cap f^{-1}V^+)) \otimes \wedge^{\text{top}}(V^+/(V^+ \cap f^{-1}V^+))^* \simeq \\ &\simeq \text{Det}_U \otimes \wedge^{\text{top}}(f^{-1}V^+/(V^+ \cap f^{-1}V^+)) \otimes \wedge^{\text{top}}(V^+/(V^+ \cap f^{-1}V^+))^* \end{aligned}$$

Local Commutator

$$\{f, g\}_{A_x}^{K_x} = \text{Norm}_{k(x)/k} \left(\frac{f^{v_x(g)}}{g^{v_x(f)}}(x) \right) \in k^*$$

and recall that

Tame Symbol

$$\Sigma_X^* \times \Sigma_X^* \ni (f, g) \longmapsto (f, g)_x := \text{class of } (-1)^{v_x(f)v_x(g)} \frac{f^{v_x(g)}}{g^{v_x(f)}} \in k(x)^*$$

Global Case

$\Sigma_X^* \hookrightarrow \mathbb{I}_X$ is a subgroup of the ideles, which acts by homotheties on the ring of adeles $\mathbb{A}_X$ and, consequently, on $\text{Gr}(\mathbb{A}_X, \mathbb{A}_X^+)$. Hence, one has:

$$1 \rightarrow k^* \rightarrow \widetilde{\Sigma_X^*} \xrightarrow{\pi_X} \Sigma_X^* \rightarrow 1$$

Theorem [Muñoz-Pablos]

The previous central extension is trivial; accordingly, it holds that:

$$\{f, g\}_{\mathbb{A}_X^+}^{\mathbb{A}_X} = 1 \quad \forall f, g \in \Sigma_X^*$$

- construct a point of $\text{Gr}(\mathbb{A}_X, \mathbb{A}_X^+)$ fixed under the action of Σ_X^* .
- construct a section of π_X which is a group homomorphism.
- Finally, if the central extension is trivial, then the commutator is also trivial.



The central extension $1 \rightarrow k^* \rightarrow \widetilde{\Sigma}_X^* \xrightarrow{\pi_X} \Sigma_X^* \rightarrow 1$ is trivial.

- The cohomology exact sequence of $0 \rightarrow \mathcal{O}_X \rightarrow \Sigma_X \rightarrow \Sigma_X/\mathcal{O}_X \rightarrow 0$ reads

$$0 \rightarrow H^0(X, \mathcal{O}_X) \rightarrow \Sigma_X \rightarrow \Sigma_X/\mathcal{O}_X \simeq \mathbb{A}_X/\mathbb{A}_X^+ \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0$$

and, since the cohomology groups are finite dimensional, it follows that

$$\Sigma_X \in \text{Gr}(\mathbb{A}_X, \mathbb{A}_X^+)$$

which is trivially invariant by the action of Σ_X^* (the relation with the Krichever map comes from $\Sigma_X = \lim_{D \subset X} H^0(X \setminus D, \mathcal{O}_X)$).

- Since Σ_X is a fixed point, $\widetilde{\Sigma}_X^*$ acts by homotheties on the fibre $\text{Det}_{\Sigma_X} \simeq k$ and, thus, the character of this action yields

$$\mu : \widetilde{\Sigma}_X^* \rightarrow k^*$$

Choose $\sigma : \Sigma_X^* \rightarrow \widetilde{\Sigma}_X^*$ a set theoretic section of π_X . One checks that

$$\Sigma_X^* \ni f \mapsto \mu(\sigma(f))^{-1} \sigma(f) \in \widetilde{\Sigma}_X^*$$

is a group homomorphism and that it is a section of π_X .



Local-Global

The explicit computation of the commutator $\{f, g\}_{\mathbb{A}_X^+}^{\mathbb{A}_X}$ can be reduced to a finite set of points of X (containing the support of the divisors of f, g) and bearing in mind how it behaves w.r.t. direct sums of vector spaces. . .

Local-Global

$$\{f, g\}_{\mathbb{A}_X^+}^{\mathbb{A}_X} = (-1)^{\sum_{x \neq y} i(f_x, A_x) i(g_y, A_y)} \prod_{x \in X} \{f_x, g_x\}_{A_x}^{K_x},$$

with $i(f, A_x) = \dim_k(A_x/A_x \cap fA_x) - \dim_k(fA_x/A_x \cap fA_x) = \deg(x) \cdot v_x(f)$.

Sampling everything together:

Reciprocity Law

$$1 = \prod_{x \in X} \text{Norm}_{k(x)/k} \left((-1)^{v_x(f)v_x(g)} \left(\frac{f^{v_x(g)}}{g^{v_x(f)}} \right)(x) \right)$$



Class Field Theory

CLASS FIELD THEORY describes the abelian extensions of a local or global field in terms of the arithmetic of the field itself.

The algebraic approach (which uses the adèle ring, the idele group, etc.) is mainly due to Artin, Weil, Chevalley, Tate, . . .

We aim at studying the situation of function fields of algebraic curves (with special focus in the case of algebraically closed base field. **[work in progress]**).

Our First Step

Multiplicative version of Cassels' results for Global Fields (i.e. finite extensions of $\mathbb{Q}$ and function fields of algebraic curves over finite fields).

$$\mathbb{A}_X := \prod'_{x \in X} K_x = \{(\alpha_x)_{x \in X} \text{ s.t. } \alpha_x \in A_x \text{ for almost all } x\}$$

$\mathbb{A}_X$ is a linearly topological space and a basis of neighborhoods of zero is:

$$U_D := \{\alpha \in \mathbb{A}_X \mid v_x(\alpha_x) \geq n_x \forall x \in X\} = \left(\prod_{x \in \text{supp } D} \mathfrak{m}_x^{n_x} \right) \times \prod_{x \notin \text{supp } D} \hat{\mathcal{O}}_{X,x},$$

with $D = \sum_{x \in X} n_x x$ is a divisor, v_x the valuation at x and $x \in X$ is closed.



Some properties

The $\mathbb{A}_X$ -topology coincides with the topology generated by the set of all subspaces $C \subset V$ s.t. $\dim(C + \mathbb{A}_X^+ / C \cap \mathbb{A}_X^+) < \infty$ as a basis of neighborhoods of zero.

Some well known properties are:

- ① The function field Σ_X is discrete in $\mathbb{A}_X$.
IDEA: $\mathbb{A}_X^+ = U_0$ and $\Sigma_X \cap \mathbb{A}_X^+ = H^0(X, \mathcal{O})$.
- ② Strong Approximation Theorem: Let $x_0 \in X$ be a rational point. Then

$$\Sigma_X \rightarrow \prod'_{x \neq x_0} K_x$$

is injective and its image is dense (w.r.t. the induced topology by $\mathbb{A}_X$).



Ideles

Since $\mathbb{A}_X$ is a topological ring, its group of invertible elements, $\mathbb{I}_X$, is endowed with the topology inherited from the map:

$$\begin{aligned}\mathbb{I}_X &\hookrightarrow \mathbb{A}_X \times \mathbb{A}_X \\ \alpha &\mapsto (\alpha, \alpha^{-1})\end{aligned}$$

A basis of open neighborhoods of $1 \in \mathbb{I}_X$ consists of:

$$V_D := \prod_{x \in \text{supp } D} (1 + \mathfrak{m}_x^{n_x}) \times \prod_{x \notin \text{supp } D} \hat{\mathcal{O}}_{X,x}^*,$$

where $D = \sum_x n_x x$ is an effective divisor on X .

To every idele we associate a divisor:

$$\mathbb{I}_X \ni \alpha \longmapsto D(\alpha) := \sum_{x \in X} v_x(\alpha_x) x \in \text{Div}(X)$$

Inspired by the classical definition of the content of $\alpha \in \mathbb{I}_X$, we introduce the following subgroup of $\mathbb{I}_X$.

$$\mathbb{I}_X^1 := \{ \alpha \in \mathbb{I}_X \text{ s.t. } \deg D(\alpha) := \sum_{x \in X} \deg(x) v_x(\alpha_x) = 0 \}$$

Remark [e.g. Cassels]

On $\mathbb{I}_X^1$, the $\mathbb{A}_X$ -topology and the $\mathbb{I}_X$ -topology do coincide.

It is enough to compare the basic neighborhoods of $\alpha = 1 \in \mathbb{I}_X^1$ w.r.t. both topologies.

Indeed, the result follows from:

$$(1 + U_D) \cap \mathbb{I}_X^1 = V_D$$

for $D = \sum_x n_x x$ an effective divisor. □

Theorem [Muñoz-Pablos-P.]

Σ_X^* is discrete and it is closed as a subgroup of $\mathbb{I}_X^1$.

Moreover, k is finite if and only if $\mathbb{I}_X^1 / \Sigma_X^*$ is compact.



A quotient of the idele group

Recall that $\mathbb{I}_X = \mathbb{A}_X^*$ acts by homotheties on $\text{Gr}(\mathbb{A}_X, \mathbb{A}_X^+)$.
Since it is commutative, the commutator yields a pairing:

$$\{, \}_{\mathbb{A}_X^+}^{\mathbb{A}_X} : \mathbb{I}_X \times \mathbb{I}_X \longrightarrow k^* .$$

Then, we define the *radical* of this pairing by:

$$R := \{ \alpha \in \mathbb{I}_X \text{ s.t. } \{ \alpha, \beta \}_{\mathbb{A}_X^+}^{\mathbb{A}_X} = 1, \forall \beta \in \mathbb{I}_X \} .$$

Theorem [Muñoz-Pablos-P.]

Let $N_x := \{ f \in k(x) \mid \text{Norm}_{k(x)/k}(f) = 1 \}$. It holds that:

$$\prod_{x \in X} ((1 + \mathfrak{m}_x) N_x) \subseteq R$$

- it is strict for k finite,
- it is an equality if k is infinite and X has a rational point.
- $R = \prod_{x \in X} (1 + \mathfrak{m}_x)$, if k is algebraically closed.

Results

We define:

$$\bar{\mathbb{I}}_X^1 := \mathbb{I}_X^1 / \prod_{x \in X} (1 + \mathfrak{m}_x)$$

and endow it with the quotient topology by the projection $\mathbb{I}_X^1 \xrightarrow{\pi} \bar{\mathbb{I}}_X^1$.

Theorem [Muñoz-Pablos-P.]

- $\bar{\mathbb{I}}_X^1$ is Hausdorff and totally disconnected.
- $\bar{\mathbb{I}}_X^1 / \Sigma_X^*$ is Hausdorff and totally disconnected.

The key is to show that a basis of neighborhoods of $1 \in \bar{\mathbb{I}}_X^1$ is given by:

$$\bar{V}_S := \prod_{x \in S} \{1\} \times \prod_{x \notin S} k(x)^* \quad \text{where } S \subset X \text{ is a finite subset.}$$



Bearing in mind that profinite is equivalent to Hausdorff, compact and totally disconnected, we recover the following well known result

Theorem [e.g. Cassels]

If k is finite, then $\bar{\mathbb{I}}_X^1 / \Sigma_X^*$ is profinite.

Thanks your for attention

