

Congreso Real Sociedad Matemática Española
Oviedo, 4 a 7 de Febrero de 2009

Equations of the moduli of Higgs pairs and infinite Grassmannian

joint with D. Hernández Serrano.

To appear in *Int. J. of Math.* `arXiv:0712.2053`

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Contents

- 1 Introduction
 - Motivation.
 - The present work
- 2 Moduli of Higgs pairs
 - A simpler case: moduli of vector bundles
 - Definition
 - Hitchin map and Krichever map
- 3 Equations of the moduli of Higgs pairs
 - Baker-Akhiezer function
 - Equations of $Higgs_X^\infty$
 - Total ramification case



Fix a curve X and a monic polynomial $p(x) \in \bigoplus_{i=0}^n H^0(X, \omega_X^{\otimes i}) x^{n-i}$.

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- **Beauville-Narasimhan-Ramanan** (1989, Abelianization program): think of a vector bundle as the direct image of a line bundle on the spectral curve and, thus, study the moduli space of vector bundles using known facts about Jacobian varieties (lots of papers; e.g. fixed determinant $\leftrightarrow$ Prym varieties).



The present work.

Further insight on the **study of the moduli space of Higgs pairs and structure of the flows on it** (recall that, the infinite Grassmannian is a natural geometrical approach to show that KP flows become linear on jacobian varieties).

We write down **equations for this moduli space**, in terms of residue identities. These can be restated in terms of differential equations for theta functions.



Krichever map for vector bundles.

Fix X projective irreducible non-singular curve over $\mathbb{C}$, $x \in X$ and $t: \widehat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \mathbb{C}[[z]]$. Let $\mathcal{U}_X^\infty$ **denote the moduli space** of data (E, ϕ) , where E is a rank n vector bundle over X and $\phi: \widehat{E}_x \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}^n$.



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The **Krichever map** is defined by:

$$\begin{aligned} Kr: \mathcal{U}_X^\infty &\rightarrow \text{Gr}(\mathbb{C}((z))^n) \\ (E, \phi) &\mapsto H^0(X - x, E) \end{aligned}$$

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$\text{Gr}(\mathbb{C}((z))^n)$ denotes the **Sato Grassmannian** associated to $\mathbb{C}((z))^n$. The set of its rational points corresponds to vector subspaces $W \subseteq \mathbb{C}((z))^n$ such that:

$$\dim_{\mathbb{C}}(W \cap \mathbb{C}[[z]]^n) < \infty \quad \text{and} \quad \dim_{\mathbb{C}}(\mathbb{C}((z))^n / W + \mathbb{C}[[z]]^n) < \infty.$$



Observe that the data (E, ϕ) can be reconstructed from $H^0(X - x, E)$. In particular, the Krichever morphism is **injective**.

Theorem (Characterization)

A point $W \in \text{Gr}(\mathbb{C}((z))^n)$ lies on $\text{Kr}(\mathcal{U}_X^\infty)$ if and only if:

$$A \cdot W \subseteq W,$$

where $A = H^0(X - x, \mathcal{O}_X)$.

Observe that $\text{Gl}(n, \mathbb{C}((z)))$ **acts** on $\text{Gr}(\mathbb{C}((z))^n)$ and also on the image of $\mathcal{U}_X^\infty$. This provides a natural way to consider flows on them.



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Moduli functor of Higgs pairs.

Let Higgs_X^∞ **denote the moduli functor of Higgs pairs** with formal extra data: the set of its rational points of Higgs_X^∞ consists of equivalence classes of triples (E, φ, ϕ) where:



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- E is a rank n vector bundle over X .
- $\varphi: E \rightarrow E \otimes_{\mathcal{O}_X} \omega_X$ is a Higgs field.
- $\phi: \hat{E}_X \xrightarrow{\sim} \hat{\mathcal{O}}_X^{\oplus n}$ (note that $t: \hat{\mathcal{O}}_{X,X} \xrightarrow{\sim} \mathbb{C}[[z]]$ induces canonically a formal trivialization $dt: \hat{\omega}_{X,X} \xrightarrow{\sim} \mathbb{C}[[z]]$).
- Compatibility condition between φ and ϕ :



Moduli functor of Higgs pairs.

There exists $M = \begin{pmatrix} 0 & \cdot & \cdot & 0 & (-1)^{n+1} a_n \\ 1 & 0 & \cdot & \cdot & (-1)^n a_{n-1} \\ 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & 0 \\ 0 & \cdot & \cdot & 0 & 1 \end{pmatrix}$, where

$(a_1, \dots, a_n) \in \mathbb{C}[[z]]^n$, making the following diagram commutative:

$$\begin{array}{ccc} \widehat{E}_x & \xrightarrow{\widehat{\phi}} & \widehat{E}_x \otimes \widehat{\omega}_{X,x} \\ \phi \downarrow \wr & & \wr \downarrow \phi \otimes dt \\ \mathbb{C}[[z]]^n & \xrightarrow{M} & \mathbb{C}[[z]]^n \end{array}$$



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Theorem

Higgs_X^∞ is representable by a $\mathbb{C}$ -scheme (not of finite type).



Hitchin map.

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$$\begin{aligned} \text{Higgs}_X &\rightarrow \bigoplus_{i=1}^n H^0(X, \omega_X^{\otimes i}) \\ (E, \psi) &\mapsto \text{coefficients of } \text{char}(\psi) \end{aligned}$$



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Formal analogue:

$$\begin{aligned} \text{Higgs}_X^\infty &\rightarrow \bigoplus_{i=1}^n \widehat{\omega}_{X,x}^{\otimes i} \stackrel{dt^{\otimes i}}{\equiv} \mathbb{C}[[z]]^n =: \mathbb{A} \\ (E, \psi, \phi) &\mapsto \text{coefficients of } \text{char}(\widehat{\psi}) \end{aligned}$$



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Observe that the functor $S \mapsto (H^0(S, \mathcal{O}_S)[[z]])^n$ is representable and, thus, it follows that $\mathbb{A}$ is a $\mathbb{C}$ -scheme.

Then, a $\mathbb{C}$ -valued point of $\mathbb{A}$ consists of a n -tuple $(a_1, \dots, a_n)$ which will be thought as a polynomial $p(T) = T^n - a_1 T^{n-1} + \dots + (-1)^n a_n$.



Krichever map.

$$\begin{array}{ccc}
 \text{Higgs}_X^\infty & \xrightarrow{\text{Kr}_{\text{Higgs}} := \text{Kr} \times \text{char}} & \text{Gr}(\mathbb{C}((z))^n) \times \mathbb{A} \\
 \searrow \text{Formal Hitchin} & & \swarrow \text{projection} \\
 & \mathbb{A} &
 \end{array}$$

It takes values in $\text{Kr}(\mathcal{U}_X^\infty) \times \mathbb{A}$ and **embeds** Higgs_X^∞ into $\text{Gr}(\mathbb{C}((z))^n) \times \mathbb{A}$ (due to the compatibility condition).

Each $p(T) \in \mathbb{A}$ endows $\mathbb{C}((z))^n$ with a $\mathbb{C}((z))$ -algebra structure:

$$\mathbb{C}((z))^n \xrightarrow{\sim} \mathbb{C}((z))[T]/p(T) =: V_p$$

So that, for each $(W, p(T)) \in \text{Gr}(\mathbb{C}((z))^n) \times \mathbb{A}$ one has $W \in \text{Gr}(V_p)$.



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Theorem (Characterization)

Let $\Omega = H^0(X - x, \omega_X)$. Then, a point $(W, p(T)) \in \text{Kr}(\mathcal{U}_X^\infty) \times \mathbb{A}$ lies on the image of Kr_{Higgs} if and only if $T(W) \subseteq W \cdot \Omega$.

This picture is very well suited for studying *fibers of the Hitchin map* as well as *group actions/flows* (work in progress).



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Given $W \in \text{Gr}(\mathbb{C}((z)))$, its **BA-function** (wave function for KP) is:

$$\psi_W(z, t) = \exp\left(-\sum_{j \geq 1} \frac{t_j}{z^j}\right) \cdot \frac{\tau_W(t_1 + z, t_2 + \frac{z^2}{2}, \dots)}{\tau_\Omega(t_1, t_2, \dots)}.$$

Theta functions allows us to write down a function on X whose expansion at x equals ψ_W .

As the parameters t 's vary, $\psi_W(z, t)$ generates W as $\mathbb{C}$ -vector space.



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$W \in \text{Gr}(V_p) \rightsquigarrow \{\psi_{1,W}(T_\bullet, t_\bullet), \dots, \psi_{r,W}(T_\bullet, t_\bullet)\}$ (r BA-functions)

Lemma

The function $\psi_{u,W}(T_\bullet, t_\bullet)$ is V_p -valued (for each $t_\bullet$) and, letting the parameter $t_\bullet$ vary, we obtain a generating system of W over $\mathbb{C}$.

Equations of Higgs_X^∞ .

We define the **adjoint BA functions of W** , $\{\psi_{1,W}^*\}_{u=1,\dots,r}$, to be the BA functions of $W^\perp$, where $W^\perp$ denotes the orthogonal of W w.r.t.

$$\begin{aligned} V_\rho \times V_\rho &\rightarrow \mathbb{C} \\ (a, b) &\mapsto \text{Res}_{z=0} (\text{Tr}(a \cdot b)) dz \end{aligned}$$

Note that if $W \in \text{Gr}(V_\rho)$ comes from a vector bundle E , then $W^\perp$ also belongs to $\text{Gr}(V_\rho)$ and it comes from $E^* \otimes \omega_X$.



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Note that, given $\Omega^{-1} \in \text{Gr}(\mathbb{C}((z)))$ and $W \in \text{Gr}(V_\rho)$, the products $\psi_{\Omega^{-1}} \psi_{v,W}$ generates $\Omega^{-1} \cdot W$ (here $\cdot$ is the $\mathbb{C}((z))$ -alg. struct. of V_ρ).



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Theorem (Equations)

The pair $(W, p(T)) \in \text{Gr}(\mathbb{C}((z))^n) \times \mathbb{A}$ lies on $\text{Kr}_{\text{Higgs}}(\text{Higgs}_X^\infty)$ if and only if the following equations are satisfied:

$$\text{Res}_{z=0} (\text{Tr}(T \cdot \psi_{u,W}^* \cdot \psi_{\Omega^{-1}} \cdot \psi_{v,W})) dz = 0 \quad 1 \leq u, v \leq r$$

Idea: use $T \cdot W \subseteq W \cdot \Omega \iff T \cdot W^\perp \subseteq (W \cdot \Omega^{-1})^\perp$.



Total ramification case.

$$\text{Higgs}_X^\infty(n) := \{(E, \varphi, \phi) \in \text{Higgs}_X^\infty \mid \pi^{-1}(x) = ny\},$$

($\pi: X_\varphi \rightarrow X$ is the spectral cover). Condition $\pi^{-1}(x) = ny$ is equivalent to the irreducibility of $p(T) = T^n - a_1 T^{n-1} + \cdots + (-1)^n a_n$.



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Given $W \subseteq V_p = \mathbb{C}((z))[T]/p(T)$, one has $\psi_W(T_\bullet, t_\bullet) \in V_p$ for each t . If we forget the $\mathbb{C}((z))$ -alg. struct., one has

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Theorem

The pair $(W, p(T)) \in \text{Gr}(\mathbb{C}((z))^n) \times \mathbb{A}$ lies on the image of $\text{Higgs}_X^\infty(n)$ if and only if $p(T)$ is irred and:

$$\sum_{0 \leq k \leq 2n-2} \sum_{i+j=k} \text{Res}_{z=0} (\psi_{W,i}^*(z) \psi_{\Omega^{-1}}(z) \psi_{W,j}(z) \text{Tr}(T^{k-1})) \frac{dz}{z^{3g}} = 0$$

$\text{Tr}(T^j)$ is a polynomial in a_i due to Newton-Girard Identities.



Thanks for your attention!

