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Solitons, Algebraic Geometry and Representation Theory

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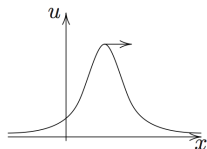


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Interplay between solitons, moduli and representations.

Solitons $\implies$ KdV equation $u_t = -6uu_x - u_{xxx}$

- where $u(t, x)$ is the elevation, t time, x space
- $u_\Lambda(x, y) = -\frac{1}{2}c \cdot p(\frac{1}{2}(\sqrt{c}(x - ct) + x_0) - \frac{1}{6}c)$ for a lattice Λ and p Weierstrass function is a solution.
- Generalizations to N -solitary waves
- Generalizations to theta functions of hyperelliptic curves and of arbitrary curves (KP)



Moduli spaces

- To points of moduli spaces one associates τ -functions of KP.
- Algebraic-geometric properties of moduli correspond to properties of τ .
- E.g. trisecant.

Representation Theory

- Virasoro algebra acts on the conformal structure of a Riemann surface and
- it also acts as Virasoro constraints (String equation, Dilaton equation, ...).
- Further research linking $\mathfrak{sl}(2)$ and **Vir** representations.

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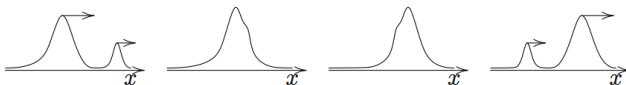
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Review of Classical Theory

A soliton is a solitary wave traveling along a shallow channel, it is described by the Korteweg-De Vries equation: $u_t = -6uu_x - u_{xxx}$.

N -soliton are N solitary waves which do not interact; e.g. 2-soliton is



Hirota provided solutions thanks to a smart substitution $u = 2\partial_x^2 \log f$ which is a reminiscence of the relation for elliptic curves (with $\Lambda = \langle 1, \Omega \rangle$, $k \in \mathbb{C}$)

$$p(z, \Omega) = -\partial_x^2 \log \theta(z + \frac{1}{2}(1 + \Omega), \Omega) + k$$

Kadomtsev and Petviashvili introduced a 2D generalization of KdV:

$$3u_{yy} = (4u_t - u_{xxx} - 6uu_x)_x$$

or, equivalently, in terms of Hirota bilinear form

$$\theta\theta_{xxxx} - 4\theta_x\theta_{xxx} + 3\theta_{xx}^2 - 3\theta_y^2 + 3\theta_{yy}\theta + 4\theta_t\theta_x - 4\theta\theta_{tx} + c\theta^2 = 0$$

Questions

Are there theta function solutions to this equation?

If so? what kind of theta functions are solutions?

Review of Classical Theory

Krichever

Theta functions of Riemann Surfaces yield solutions.

see also works of [Burchall-Chaundy, Mumford, ...]

Shiota

KP equation characterizes jacobian theta functions among theta of p.p.a.v.
It is analogous of the Schottky problem.

Note also the characterization in terms of a trisecant.

Sato-Sato

They showed that the set of τ -functions of the KP hierarchy correspond bijectively to the set of points of an infinite Grassmann manifold.

Many relevant contributions

- Mumford: rings generated by two commuting differential operators;
- Segal-Wilson: analytical construction of infinite Grassmannian;
- Date-Jimbo-Kashiwara-Miwa (Kyoto school): pseudodifferential operators,
- Mulase: finite dimensional orbits of KP flows;



Algebraic Formalism. Grassmannian.

Theorem [Álvarez-Muñoz-P.]

There exists a scheme $\text{Gr} := \text{Gr}(\mathbb{C}((z)))$ whose set of rational points is

$$\text{Gr} := \{U \subset \mathbb{C}((z)) \text{ s.t. } \pi_U : U \rightarrow \mathbb{C}((z))/\mathbb{C}[[z]] \text{ has finite dim ker and coker}\}$$

Let Det be the determinant line bundle on Gr ; whose fiber at U is

$$\text{Det}_U := \wedge \ker(\pi_U) \otimes (\wedge \text{coker } \pi_U)^*$$

$$\det(\pi_U) \in H^0(\text{Gr}, \text{Det}^*)$$

The linear group of $\mathbb{C}((z))$ acts on the Grassmannian; in particular, the formal scheme $\Gamma := \{\gamma = \exp(\sum t_i z^{-i})\}$ acts by homotheties $\gamma : \text{Gr} \rightarrow \text{Gr}$:

$$U \mapsto \gamma \cdot U = U(t) := \exp(\sum t_i z^{-i}) \cdot U$$

$$\gamma^* \text{Det} \simeq \text{Det} \quad \forall \gamma \in \Gamma \quad \implies \quad \tau_U(t) := \frac{\det(\pi_{U(t)})}{\det(\pi_U)} \in \mathcal{O}_\Gamma = \mathbb{C}[[t_1, t_2, \dots]]$$

Further, the Det bundle yields the Plücker embedding

$$\text{Gr } \mathbb{C}((z)) \xrightarrow{\text{Plücker}} \mathbb{P}H^0(\text{Gr}, \text{Det}^*)^* \simeq \mathbb{P}\mathbb{C}[[t_1, t_2, \dots]]$$

The Plücker relations for U are equivalent to the KP hierarchy for $\tau_U(t)$.



Algebraic Formalism. Moduli of Curves.

Regarding moduli spaces, note that there is an (functor) homomorphism

$$\begin{aligned}\bar{\mathcal{M}} &:= \{ \text{triples } (X, p, \phi) \} \rightarrow \text{Gr} \\ (X, p, \phi) &\longmapsto H^0(X - p, \mathcal{O}_X) \subset (\hat{\mathcal{O}}_{X,p})_{(0)} \simeq \mathbb{C}((z))\end{aligned}$$

whose image consists of the set $\{A \in \text{Gr} \text{ s.t. } A \text{ is } \mathbb{C}\text{-algebra}\}$.

Being a subalgebra is a locally closed condition and $\bar{\mathcal{M}}$ is a subscheme of Gr .

This idea can be generalized to a bunch of moduli spaces.

The explicit relation of τ and θ follows from Krichever.

Given (X, p, z) with period matrix Ω_X .

Take 1-forms, η_n , holomorphic at $X - p$ and $\eta_n = d(z^{-n}) + O(1)$ at p .

Consider the quadratic form $Q(t) = \sum_{n,m \geq 1} q_{nm} t_n t_m$ where

$$\int^x \eta_n = z^{-n} - 2 \sum_{m \geq 1} q_{nm} \frac{z^m}{m}$$

and the $g \times \infty$ -matrix A whose j -th row is the expansion of ω_j at p .

$$\tau_X(t) = \exp Q(t) \Theta(A(t); \Omega_X) \quad t = (t_1, t_2, \dots)$$

Applications.

$G = \text{Aut}_{\mathbb{C}\text{-alg}} \mathbb{C}((z))$ can be endowed with a scheme structure.

The canonical action on Gr preserves Det and $\bar{\mathcal{M}}$ and, for (X, p, ϕ) is as follows

$$(g, X) \mapsto X^g := ((X - D^\circ) \sqcup D) / \sim$$



Similar arguments can be applied to the modified Krichever map:

$$K_\beta : (X, p, \phi) \mapsto H^0(X - p, \omega_X^{\otimes \beta})$$

Let Λ_β be the torsor defined by G on $\bar{\mathcal{M}}$ via K_β .

$\text{Lie}(\Lambda_1) \simeq \mathbf{Vir}$, the Virasoro algebra since the correspondence of $z(1 + \epsilon z^n) \in G$ with $L_n := -z^{n+1}\partial_z \in \text{Der } \mathbb{C}((z))$ satisfies $[L_m, L_n] = (m - n)L_{m+n}$.

The action is now the Kodaira-Spencer map / Virasoro Uniformization and

$$0 \rightarrow H^0(X - p, \mathbb{T}_X) \rightarrow \mathbf{Witt} = \mathbb{C}((z))\partial_z \rightarrow T_{(X, p, \phi)}\bar{\mathcal{M}} = \varprojlim_n H^1(X, \mathbb{T}_X(-np)) \rightarrow 0$$

and L_n changes the formal coordinate z for $n \geq 0$, L_{-1} shifts (infinitesimally) the point p , L_n may change the conformal structure for $n \leq -2$.

Theorem (Muñoz-P., formal Mumford's isomorphism, applications to bosonic string)

$\Lambda_\beta \simeq K_\beta^* \text{Det}$, it is isomorphic to the pullback of the β -th Hodge bundle and

$$\Lambda_\beta \simeq \Lambda_1^{6\beta^2 - 6\beta + 1}$$

Partition Function and Virasoro Constraints

Let $\bar{\mathcal{M}}_{g,n}$ be the compactification of moduli stack of n punctured genus g R.S.
Let L_i be the line bundle on $\bar{\mathcal{M}}_{g,n}$ whose fiber at $(X, x_1, \dots, x_n)$ is the $\mathbb{T}_{X, x_i}^*$.

Theorem [Witten-Kontsevich, generating function of intersection indexes]

$$F(t_0, t_1, \dots) := \sum \langle \tau_0^{k_0} \tau_1^{k_1} \dots \rangle \prod_{i \geq 0} \frac{t_i^{k_i}}{k_i!} := \sum \left(\int_{\bar{\mathcal{M}}_{g,n}} \prod c_1(L_i)^{k_i} \right) \prod_{i \geq 0} \frac{t_i^{k_i}}{k_i!}$$

- F is the partition function in the standard matrix model theory.
- $\exp(F)$ in $T_{2i+1} := \frac{t_i}{(2i+1)!!}$ is a τ -function for the KdV hierarchy
- F satisfies the Virasoro constraints $L_n F = 0$ for $n \geq -1$.

L_{-1} is the string equation, L_0 the dilaton equation, and L_n is given by

$$\begin{aligned} L_n := & - \left(\frac{(2n+3)!!}{2^{n+1}} \right) \partial_{t_{n+1}} + \sum_{i \geq 0} \left(\frac{(2i+2n+1)!!}{(2i-1)!! 2^{n+1}} \right) t_i \partial_{t_{i+n}} \\ & + \frac{\lambda^2}{2} \sum_{i=0}^{n-1} \left(\frac{(2i+1)!! (2n-(2i+1))!!}{2^{n+1}} \right) \partial_{t_i} \partial_{t_{n-i-1}} \end{aligned}$$

Problem

Find common solutions to KdV and Virasoro.

Action of the Lie Algebra

$$\rho : G \curvearrowright \text{Gr} \implies \begin{array}{c} 0 \rightarrow \mathbb{C}^* \rightarrow \tilde{G} := \left\{ (\tilde{g}, g) \text{ s.t. } \begin{array}{ccc} \text{Det} & \xrightarrow{\tilde{g}} & \text{Det} \\ \downarrow & & \downarrow \\ \text{Gr} & \xrightarrow[g]{\sim} & \text{Gr} \end{array} \right\} \rightarrow G \rightarrow 0 \\ \\ \tilde{\rho} : \tilde{G} \curvearrowright H^0(\text{Gr}, \text{Det}^*) \simeq \mathbb{C}[[t_1, t_2, \dots]] \end{array}$$

and $\tilde{G}$ is trivial iff the associated 2-cocycle ([Gelfand-Fuchs]) is trivial.
For $U \in \text{Gr}$, at the level of tangent spaces,

$$\text{Lie } G = \mathbf{Witt} \rightarrow T_U \text{Gr} = \text{Hom}(U, \mathbb{C}((z))/U)$$

$$L \mapsto U \hookrightarrow \mathbb{C}((z)) \xrightarrow{\rho(L)} \mathbb{C}((z)) \rightarrow \mathbb{C}((z))/U$$

and the above construction yields the Virasoro constraints

$$\begin{aligned} \text{Lie } \tilde{G} = \mathbf{Vir} &\rightarrow \mathbb{C}[[t_1, t_2, \dots]] \\ L &\mapsto \tilde{\rho}(L)\tau_U(t) \end{aligned}$$

- $\rho(L)(U) \subseteq U$ if and only if $\tilde{\rho}(L)\tau_U(t) = 0$
- $G = \text{Aut}_{\mathbb{C}\text{-alg}} \mathbb{C}((z)) \implies \text{Lie } G = \mathbf{Witt} := \langle \{L_n | n \in \mathbb{Z}\} \rangle$, $\text{Lie } \tilde{G} = \mathbf{Vir}$
- For any representation of $\mathbf{Vir}^+ := \langle \{L_n | n \geq -1\} \rangle$, the cocycle is trivial

Translating the Problem

Now, we can go back and forth from Gr to the Boson Fock Space:

Action on $\mathbb{C}((z))$	Action on $\mathbb{C}[[t_1, t_2, \dots]]$
z^n with $n \geq 1$	nt_n
z^{-n} with $n \geq 1$	∂_{t_n}
1	0
$z^n(z\partial_z + \frac{n+1}{2})$ with $n \geq 1$	$\frac{1}{2} \sum_{i=1}^{n-1} i(n-i)t_i t_{n-i} + \sum_{i=1}^{\infty} (n+i)t_{n+i}\partial_{t_i}$
$z^{-n}(z\partial_z + \frac{1-n}{2})$ with $n \geq 1$	$\frac{1}{2} \sum_{i=1}^{n-1} \partial_{t_i} \partial_{t_{n-i}} + \sum_{i=1}^{\infty} it_i \partial_{t_{n+i}}$
For $U \subset \mathbb{C}((z))$	For $\tau_U(t) \in \mathbb{C}[[t_1, t_2, \dots]]$
$U \in \text{Gr}$	KP hierarchy
$z^{-2}U \subseteq U$	$\partial_{t_{2i}}\tau_U(t) = 0$ for all i (KdV, provided KP)
$\rho(L)(U) \subseteq U$	$\tilde{\rho}(L)\tau_U(t) = 0$

This is ok for one element, but

For a group G acting on $\mathbb{C}((z))$, one has to check that the extension $\tilde{G}$ splits.

Representations of Virasoro

Theorem (P.)

There exists a bijection

$$\left\{ \begin{array}{l} \rho : \mathbf{Vir}^+ \rightarrow \text{Diff}^1(\mathbb{C}((z))) \\ \text{of Lie alg. such that } \rho \neq 0 \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{triples } (h(z), c, b(z)) \text{ such that} \\ h'(z) \neq 0, c \in \mathbb{C}, b(z) \in \mathbb{C}((z)) \end{array} \right\}$$

which is explicitly given by:

$$\rho(L_i) = \frac{-h(z)^{i+1}}{h'(z)} \partial_z - (i+1)c \cdot h(z)^i + \frac{h(z)^{i+1}}{h'(z)} b(z)$$

Theorem (P.)

Let $U \subseteq \mathbb{C}((z))$. If $\rho(L_i)(U) \subseteq U$ for all $i \geq -1$, then

$$A_U := \{f \in \mathbb{C}((z)) \text{ such that } f \cdot U \subseteq U\} = \mathbb{C}[h]$$

$$A_U^1 := \{D \in \text{Diff}^1 \mathbb{C}((z)) \text{ such that } D(U) \subseteq U\} = \mathbb{C}[h] \oplus \text{Im } \rho$$

If $h(z) = z^{-2}$, $\text{Res}_{z=0} b(z) = \frac{3}{2}$ and $\dots$, then the central extension $\widetilde{A}_U^1$ splits.



Existence of Solutions

Bearing in mind the previous results, it suffices to construct invariant subspaces.

Theorem (P.)

Let ρ be given. There is an auxiliary function $v(z)$ (depending on the choice of a solution of the Airy equation) such that the subspace

$$\mathcal{U} := \mathbb{C}[h(z)] \otimes \langle 1 \otimes v(z), \rho^v(L_{-1})(1) \otimes v(z) \rangle \subset \mathbb{C}((z)) \otimes v(z)$$

fulfills

- ① $\mathcal{U}$ belongs to the Sato Grassmannian of $\mathbb{C}((z)) \otimes v(z)$;
- ② $z^{-2}\mathcal{U} \subset \mathcal{U}$;
- ③ $\rho^v(L_i)\mathcal{U} \subseteq \mathcal{U}$ for $i \geq -1$.

Idea of proof:

- ① It follows from the choice of $v(z)$.
- ② By the very construction $z^{-2} \in \mathbb{C}[h(z)] \subseteq A_{\mathcal{U}}$.
- ③ For $\rho^v(L_i)$ consider:
 - $i = -1$: note that $\rho^v(L_{-1})^2(u)$ is a linear combination of u and $\rho^v(L_{-1})(u)$ with coefficients in $\mathbb{C}[h(z)]$ (this fact relies on the Airy equation).
 - for $i \geq 0$: note that

$$\rho^v(L_i) = h(z)^i(h(z)\rho^v(L_{-1}) - (i+1)c)$$



Uniqueness

Proposition (P.)

Given ρ , let $\mathcal{U}_1, \mathcal{U}_2$ be two subspaces corresponding to two different choices of $v(z)$. Then, they differ in the exponential of a linear function; i.e.,

$$\partial_{t_{2i+1}} \partial_{t_{2j+1}} \log \tau_{\mathcal{U}_1} = \partial_{t_{2i+1}} \partial_{t_{2j+1}} \log \tau_{\mathcal{U}_2} \quad \forall i, j, \geq -1$$

Stating the properties of $\mathcal{U}$ in terms of its τ -function, one obtains that

- $\tau_{\mathcal{U}}(t)$ satisfies the KdV hierarchy
- $\tau_{\mathcal{U}}(t)$ satisfies the Virasoro constraints $\rho^v(L_i)\tau_{\mathcal{U}}(t) = 0$ for $i \geq -1$.

Theorem (P.)

The Virasoro constraints admit a unique solution (provided KdV and up to the exponential of a linear function).

A function can solve only one set of Virasoro constraints at most.

In other words, let (ρ_j, τ_j) be given such that $\rho_j(L_i)\tau_j = 0$ for $j = 1, 2$.

The following conditions are equivalent

- $\partial_{t_{2i+1}} \partial_{t_{2j+1}} \log \tau_1 = \partial_{t_{2i+1}} \partial_{t_{2j+1}} \log \tau_2$;
- ρ_1 and ρ_2 induce the same set of differential equations.



Geometric interpretation of the solution

Theorem (P.)

Given ρ with $h(z) = z^{-2}$.

We associate to ρ a pair $(\mathcal{E}, \nabla)$, consisting of a vector bundle of rank 2 over the projective line, and a connection on it.

The proof relies on the Krichever construction. Let $\mathcal{U}$ be an invariant subspace (with the previous notations).

- $\mathbb{P}_1$ is obtained by gluing $\text{Spec } A_{\mathcal{U}} = \text{Spec } \mathbb{C}[h(z)]$ with the valuation given by $h(z)^{-1}$.
- $\mathcal{E}$ is the rank 2 vector bundle on $\mathbb{P}_1$ defined by $\mathcal{U}$.
- ∇ is a the connection on the bundle $\mathcal{E}$ defined by the map

$$\begin{aligned} \mathcal{U} \otimes_{\mathbb{C}} \mathbb{C}[h(z)] &\rightarrow \mathcal{U} \otimes_{\mathbb{C}} \Omega_{\mathbb{C}[h(z)]/\mathbb{C}} \\ f \otimes a &\mapsto \rho(L_{-1})f \otimes a \, dh - f \otimes da \end{aligned}$$

(because of $\rho(L_{-1})\mathcal{U} \subseteq \mathcal{U}$ and the properties of ρ)

Remark Given $(\mathbb{P}_1, \rho, \bar{z}, \mathcal{E}, \nabla)$, note $\text{Der}(\mathcal{O}_{\mathbb{P}_1})_{(0)} \simeq \mathbf{Witt} = \langle \{\bar{z}^{n+1}\partial_{\bar{z}}\} \rangle$, and

$$\bar{z}^{n+1}\partial_{\bar{z}} \longmapsto \nabla_{\bar{z}^{n+1}\partial_{\bar{z}}} \quad \text{i.e.} \quad \nabla_{\bar{z}^{n+1}\partial_{\bar{z}}}(f) := \langle \nabla(f), \bar{z}^{n+1}\partial_{\bar{z}} \rangle$$

is a representation of **Witt**.



Representations of $\mathbf{Vir}^+$ in mathematical physics

Reps of $\mathbf{Vir}^+$ arise in math-phys as systems of differential equations. E.g.

- Partition Function of 2D quantum gravity: Dijkgraaf-Verlinde-Verlinde, Witten-Kontsevich, Givental, ...
- Virasoro Conjecture (topological sigma model, Frobenius manifolds, ...): Eguchi-Hori-Xiong, Katz, Dubrovin, ...
- Topological Recursion Relations (Eynard-Orantin): Mulase-Safnuk, Liu-Xu, ...

Let us write down the operators on $\mathbb{C}[q_0, q_1, \dots]$ corresponding to the last case:
 $\mathbf{m} := (m_1, m_2, \dots)$ non-negative integers with $m_i = 0, \forall i \gg 0$, $\mathbf{s} := (s_1, s_2, \dots)$ variables.

$$|\mathbf{m}| := \sum_{i \geq 1} i m_i \quad \|\mathbf{m}\| := \sum_{i \geq 1} m_i \quad \mathbf{m}! := \prod_{i \geq 1} m_i! \quad \mathbf{s}^{\mathbf{m}} := \prod_{i \geq 1} s_i^{m_i}$$

$$\begin{aligned} \bar{L}'_n(\mathbf{s}) := & -\frac{1}{2} \sum_{\mathbf{m}} \frac{(-1)^{\|\mathbf{m}\|}}{\mathbf{m}!(2|\mathbf{m}|+1)!!} \mathbf{s}^{\mathbf{m}} \partial_{q_{|\mathbf{m}|+n+1}} + \sum_{i=0}^{\infty} (i + \frac{1}{2}) q_i \partial_{q_{i+n}} + \\ & + \frac{1}{2} \sum_{i=1}^n \partial_{q_{i-1}} \partial_{q_{n-i}} + \frac{q_0^2}{4} \delta_{n,-1} + \frac{1}{16} \delta_{n,0} \end{aligned}$$

They fulfill $[\bar{L}'_i(\mathbf{s}), \bar{L}'_j(\mathbf{s})] = (i-j)\bar{L}'_{i+j}(\mathbf{s})$ for $i, j \geq -1$.



Representations of Virasoro and $\mathfrak{sl}(2)$

The previous diff. operators on $\mathbb{C}[q_0, q_1, \dots]$ come from diff. operators on $\mathbb{C}((z))$ via bosonization; more precisely,

Theorem (P.)

$\rho'_s(L_i) := \bar{L}'_i(\mathbf{s})$ is attached to the data $(h(z) = z^{-2}, c = \frac{1}{2}, b_s(z))$ where:

$$b_s(z) := - \sum_{\mathbf{m}} \frac{(-1)^{\|\mathbf{m}\|}}{\mathbf{m}!(2|\mathbf{m}|+1)!!} \mathbf{s}^{\mathbf{m}} z^{-2|\mathbf{m}|-4} - \frac{3}{2} z^{-1}$$

Set a Chevalley basis $\{e, f, h\}$ of $\mathfrak{sl}(2)$, $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$.

$\mathfrak{sl}(2) \simeq \langle L_{-1}, L_0, L_1 \rangle \subset \mathbf{Vir}^+$ by sending $f \mapsto L_{-1}$, $h \mapsto -2L_0$, $e \mapsto -L_1$

A representation $\rho : \mathbf{Vir}^+ \rightarrow \text{Diff}^1 \mathbb{C}((z))$ is associated to $(h(z), c, b(z))$

$$\rho(L_i) = \frac{-h(z)^{i+1}}{h'(z)} \partial_z - (i+1)c \cdot h(z)^i + \frac{h(z)^{i+1}}{h'(z)} b(z)$$

and it is thus determined by $\{\rho(L_{-1}), \rho(L_0), \rho(L_1)\}$; i. e. by $\rho|_{\mathfrak{sl}(2)}$.

Question

How general is this fact? That is, when ρ is determined by $\rho|_{\mathfrak{sl}(2)}$?
It holds true for infinite dimensional simple weight representations.

Criterion

Theorem (P. - Tejero)

Let M be an $\mathfrak{sl}(2)$ -module and σ the action. Let $T \in \text{End}(M)$. Define ρ by

$$\rho|_{\mathfrak{sl}(2)} = \sigma \quad \text{and} \quad \rho(L_{2+i}) := \frac{1}{i!} \text{ad}(\sigma(e))^i(T) \quad i \geq 0$$

Then, ρ defines a compatible $\mathbf{Vir}^+$ action, if and only if

$$\text{ad}(\sigma(f))(T) = 3\sigma(e)$$

$$\text{ad}(\sigma(h))(T) = 4T$$

$$[T, \rho(L_{2+i})] = -i\rho(L_{4+i}) \quad \forall i > 0$$

The infinite set of conditions given above may *collapse* in special cases:

Let $\mathcal{M}$ be the category of weight $\mathfrak{sl}(2)$ -modules such that a) the Casimir C acts by $\tau \in \mathbb{C}$; and, b) $\tau \neq (\mu + 1)^2$ for every weight μ . Let $M \in \mathcal{M}$ and T, ρ be as above. Then, ρ yields a compatible $\mathbf{Vir}^+$ action, if and only if

$$\text{ad}(\sigma(f))(T) = 3\sigma(e)$$

$$\text{ad}(\sigma(h))(T) = 4T$$

$$\text{ad}(\sigma(c))(T) = 0$$

$$[T, \text{ad}(\sigma(e))(T)] = -\frac{1}{6} \text{ad}(\sigma(e))^3(T)$$



Simple weight modules

An $\mathfrak{sl}(2)$ -module V is called weight if $V = \bigoplus_{\lambda \in \mathbb{C}} \ker(h - \lambda)$.

Similarly, an $\mathbf{Vir}^+$ -module V is called weight if $V = \bigoplus_{\lambda \in \mathbb{C}} \ker(L_0 - \lambda)$.

There are three types of infinite dimensional simple weight $\mathfrak{sl}(2)$ -modules.

- *highest weight module* $M(\lambda)$ for $\lambda \in \mathbb{C} \setminus \{0, 1, 2, \dots\}$. Let $M(\lambda)$ be the vector space generated by $\{w_i | i = 0, 1, 2, \dots\}$ with the $\mathfrak{sl}(2)$ -action

$$f(w_i) = w_{i+1}$$

$$h(w_i) = (\lambda - 2i)w_i$$

$$e(w_i) = \begin{cases} i(\lambda - i + 1)w_{i-1} & \text{for } i \neq 0 \\ 0, & \text{for } i = 0 \end{cases}$$

$\{\lambda, \lambda - 2, \lambda - 4, \dots\}$ is the set of weights. We say that λ is the highest.

- *lowest weight module* $\bar{M}(\lambda)$. Let $\lambda \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$.
- *dense modules* $\mathbb{V}(\xi, \tau)$. For $\xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$ with $\tau \neq (\mu + 1)^2$.

$\mathbb{V}(\xi, \tau)$ is the vector space generated by $\{v_\mu | \mu \in \xi\}$ endowed the action

$$f(v_\mu) = v_{\mu-2} \quad h(v_\mu) = \mu v_\mu \quad e(v_\mu) = \frac{1}{4}(\tau - (\mu + 1)^2)v_{\mu+2}$$



Theorem (P.-Tejero)

Every infinite dimensional simple weight $\mathfrak{sl}(2)$ -module admits a compatible structure of $\mathbf{Vir}^+$ -module.

We obtain explicit expressions for these actions.

For the case of *highest weight $\mathfrak{sl}(2)$ -module $M(\lambda)$* for $\lambda \in \mathbb{C} \setminus \{0, 1, 2, \dots\}$:

$$f(w_i) = w_{i+1}$$

$$h(w_i) = (\lambda - 2i)w_i$$

$$e(w_i) = \begin{cases} i(\lambda - i + 1)w_{i-1} & \text{for } i \neq 0 \\ 0, & \text{for } i = 0 \end{cases}$$

there is a unique structure of $\mathbf{Vir}^+$ -module:

$$\rho(L_i)w_j := \begin{cases} -\frac{1}{2}(2(i-j) + (i+1)\lambda) \frac{\Gamma(j+1)}{\Gamma(j-i+1)} w_{j-i} & \text{for } j-i \geq 0 \\ 0 & \text{for } j-i < 0 \end{cases}$$

Γ being the Gamma-function.



Remark:

- ① The previous result can be generalized for **Witt** and **Vir**.
- ② There are infinite dimensional non simple weight $\mathfrak{sl}(2)$ -modules with no compatible structure of **Vir**⁺-module.

Although there are simple non-weight $\mathfrak{sl}(2)$ -modules with no compatible structure of **Witt**⁺-module, there are evidences for the following

Conjecture (P. - Tejero)

There is a canonical correspondence between simple $\mathfrak{sl}(2)$ -modules and simple **Vir**⁺-modules.

Thanks for your attention!

The main references for the topics of the talk are the papers by Burchnall-Chaundy, Date-Jimbo-Kashiwara-Miwa, Dijgraaf-Verlinde-Verlinde, Dubrovin, Givental, Kac-Schwarz, Krichever, Kontsevich, Mathieu, Mulase, Mulase-Safnuk, Mumford, Sato-Sato, Segal-Wilson, Shiota, Witten, ... (alphabetical order)

The precise statements of the original results stated in this talk can be found in

- Álvarez, A.; Muñoz, J. M.; -, “The algebraic formalism of soliton equations over arbitrary base fields”, Aportaciones Mat. Investig., 13, Soc. Mat. Mex. 1998
- Muñoz, J. M.; -, “Automorphism group of $k((t))$: applications to the bosonic string”, Comm. Math. Phys. 216 (2001), no. 3, 609–634.
- -, “Algebro-geometric solutions of the string equation”, arXiv:1110.0729.
- -, ‘Representations of the Witt Algebra and $\text{Gl}(n)$ -Opers”, Letters in Mathematical Physics (2013), Volume 103.
- -, C Tejero-Prieto, “Extending representations of $\mathfrak{sl}(2)$ to Witt and Virasoro algebras”, work in progress.

