

Non-abelian theta functions:
addition formulae and
multicomponent KP hierarchy

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Objectives of the talk:

- Generalize Fay's addition formula for theta functions of Jacobians for theta functions of moduli spaces of vector bundles.
- Show that theta functions of moduli spaces of vector bundles give rise to tau functions of the multicomponent KP hierarchy (similarly to the jacobian case).

These results are to be found in:

- *Algebraic Solutions of the Multicomponent KP Hierarchy*, To appear in Journal of Geom. and Phys. (math. AG/990769).
- *Addition Formulae for Non-abelian Theta Functions*, joint work with E. Gómez González, math.AG/0008005.

Plan of the talk:

1. Quick overview of the jacobian case.
2. Addition formula for non-abelian thetas.
3. Infinite grassmannians: tau and Baker functions.
4. Multicomponent KP hierarchy.
5. Relation between non-abelian thetas and the multicomponent KP hierarchy.

Notation:

- C will be a complete smooth irreducible curve over the complex numbers.
- g will denote the genus of C .
- J_d will be the jacobian of degree d .
- $\Theta_{g-1} := \{L \in J_{g-1} \mid h^0(L) \neq 0\}$ is the canonical polarization: the theta divisor.
- θ is the theta function: the unique section of $\mathcal{O}_{J_{g-1}}(-\Theta)$.
- $E(\cdot, \cdot)$ will be the prime form (it's the unique section of $\mathcal{O}_{C \times C}(\Delta)$).
- for a bundle L and a divisor D , we define $L(D) := L \otimes \mathcal{O}_C(D)$.

§1.1: Fay's addition formula

$$L \in J_{g-1} - \Theta$$

$$D \in \text{Div}^{g+n-1}(C) \text{ such that } h^0(L(D)) = m$$

$$\{s_1, \dots, s_m\} \text{ a basis of } H^0(L(D))$$

Then, for $B = \sum_{i=1}^{m-n} b_i$ generic, it holds that:

$$\theta\left(L(D + B - \sum_{i=1}^m x_i)\right)$$

and:

$$\frac{\prod E(b_i, x_j)}{\prod E(x_i, x_j)} \begin{vmatrix} s_1(x_1) & \dots & s_1(x_m) \\ \vdots & & \vdots \\ s_m(x_1) & \dots & s_m(x_m) \end{vmatrix}$$

have the same divisor as holomorphic functions of $(x_1, \dots, x_m) \in C^m$.

Important consequences are:

- the trisecant identity,
 - θ gives solutions to the KP equation.
- (Both facts are related to the Schottky problem).

§1.2: KP hierarchy

$U \in \text{Gr}^0 \mathbf{C}((z))$ such that $z^m U \cap \mathbf{C}[[z]]$ has a basis $f_1, \dots, f_m$.

Its tau function satisfies ($m \gg 0$):

$$\tau_U(x_1, \dots, x_m) = \prod (x_i - x_j)^{-1} \cdot \begin{vmatrix} f_1(x_1) & \dots & f_1(x_m) \\ \vdots & & \vdots \\ f_m(x_1) & \dots & f_m(x_m) \end{vmatrix}$$

This implies that such functions are tau functions for the KP hierarchy.

§1.3: Relation

Is this a mere coincidence? **NO**. The former formula is just an **instance** of the latter.

Fix C and $p \in C$. Consider the morphisms:

$$\begin{array}{c} \{(L, \alpha)\} \xrightarrow{\quad K \quad} \text{Gr}^0 \mathbf{C}((z)) \\ \pi \downarrow \\ J_{g-1} \end{array}$$

then:

$$\pi^* \text{Det} = K^* \mathcal{O}_J(\Theta)$$

§2: Addition formula for non-abelian thetas

§2.1: Non-abelian theta functions

$\mathcal{U}(r, d)$ will denote the moduli space of semistable vector bundles of rank r and degree d .

On $\mathcal{U}(r, r(g-1))$ there is a polarization:

$$\Theta_r := \{M \text{ such that } h^0(M) \neq 0\}$$

Since $h^0(\mathcal{O}(-\Theta_r)) = 1$, let θ_r be a generator.

For arbitrary d , choose a v.b. F such that:

$$\otimes F : \mathcal{U}(r, d) \hookrightarrow \mathcal{U}(\bar{r}, \bar{r}(g-1))$$

and define a polarization $\Theta_F := (\otimes F)^{-1}(\Theta_{\bar{r}})$.

A v.b. M on $C \times S$ has an associated morphism:

$$\phi_M : S \rightarrow \mathcal{U}(r, r(g-1))$$

Then:

$$\phi_M^* \mathcal{O}(\Theta_r) \simeq \text{Det}(R^\bullet p_{S,*} M)^*$$

since for any divisor D of degree big enough:

$$\begin{aligned} 0 \rightarrow p_{S,*} M \rightarrow p_{S,*} M(D) &\xrightarrow{\rho} \\ \rightarrow p_{S,*} (M \otimes \mathcal{O}_D(D)) \rightarrow R^1 p_{S,*} M &\rightarrow 0 \end{aligned}$$

§2.2: Isomorphism of bundles

Given M on $C \times S$ as before, consider the "addition morphism":

$$\begin{aligned} \alpha_n : C^{2n} \times S &\longrightarrow \mathcal{U}(r, r(g-1)) \\ ((x_1, y_1), \dots, (x_n, y_n)) &\longmapsto M(\sum x_i - \sum y_i) \end{aligned}$$

It is the morphism associated to the v.b.

$M(\sum_{i \text{ odd}} \Delta_{0i} - \sum_{i \text{ even}} \Delta_{0i})$ on $C \times (C^{2n} \times S)$.

Theorem. *There is an isomorphism of line bundles on $C^{2n} \times S$:*

$$\begin{aligned} \alpha_n^* \mathcal{O}(\Theta_r) &\simeq \phi_M^* \mathcal{O}(\Theta_r) \otimes \mathcal{O} \left(\sum_{i+j=\text{odd}} \Delta_{ij} - \sum_{i+j=\text{even}} \Delta_{ij} \right)^{\otimes r} \otimes \\ &\otimes \left(\bigotimes_{i \text{ odd}} \pi_i^* \wedge M^* \otimes p_i^* \omega_C^{\otimes r} \right) \otimes \left(\bigotimes_{i \text{ even}} \pi_i^* \wedge M \right) \end{aligned}$$

(sums involve only i, j with $i < j$) where:

- Δ_{ij} is the subscheme of C^{2n} where the i -th and the j -th components coincide,
- p_i (resp. π_i) is the projection from $C^{2n} \times S$ onto the i -th copy C_i of C , (resp. to $C_i \times S$),
- $\wedge M$ denotes the exterior algebra of highest degree of a locally free module M , and
- ω_C is the canonical sheaf of C .

§2.3: Addition formula

$$M \in \mathcal{U}(r, d) - \Theta_F$$

$$D \in \text{Div}^{g+n-1}(C) \text{ such that } h^0(M \otimes F(D)) = m \cdot \bar{r}$$

$$\{s_i = (s_i^1, \dots, s_i^{\bar{r}}) \mid i = 1, \dots, m \cdot \bar{r}\} \text{ a basis of } H^0(M \otimes F(D))$$

Then, for $B = \sum_{i=1}^{m-n} b_i$ generic, it holds that:

$$\frac{\theta_F(M(D + B - \sum_{i=1}^m x_i))}{\theta_F(M)} = \lambda \frac{\prod E(b_i, x_j)^{\bar{r}}}{\prod E(x_i, x_j)^{\bar{r}}} \cdot |S|$$

as holomorphic functions of $(x_1, \dots, x_m) \in C^m$, where λ is a non-zero constant (depending only on C) and S is the following matrix:

$$\begin{pmatrix} s_1^1(x_1) & \dots & s_1^{\bar{r}}(x_1) & \dots & s_1^1(x_m) & \dots & s_1^{\bar{r}}(x_m) \\ \vdots & & & & & & \vdots \\ s_{m\bar{r}}^1(x_1) & \dots & s_{m\bar{r}}^{\bar{r}}(x_1) & \dots & s_{m\bar{r}}^1(x_m) & \dots & s_{m\bar{r}}^{\bar{r}}(x_m) \end{pmatrix}$$

§2.4: Applications to cyclic coverings

Using the addition formula, Fay studied the theta function of a Prym variety. Some of them can be generalized.

Example. *Let $\gamma : \tilde{C} \rightarrow C$ a non-ramified cyclic covering of degree n . Let $\tilde{M}$ be a semistable v.b. on $\tilde{C}$ such that $\gamma_*\tilde{M}$ is semistable.*

Then for $(x_1, y_1, \dots, x_m, y_m) \in C^{2m}$:

$$\begin{aligned} \frac{1}{\theta_F(\gamma_*\tilde{M})} \theta_F\left((\gamma_*\tilde{M})\left(\sum_{i=1}^m x_i - \sum_{i=1}^m y_i\right)\right) &= \\ &= \lambda \left(\frac{1}{\tilde{\theta}_{\tilde{r}}(\tilde{M})} \tilde{\theta}_{\tilde{r}}\left(\tilde{M} \otimes \gamma^* \mathcal{O}\left(\sum_{i=1}^m x_i - \sum_{i=1}^m y_i\right)\right) \right)^n \end{aligned}$$

where λ is a non-zero constant and $F = (\gamma_*\mathcal{O}_{\tilde{C}})^*$.

§3: Infinite grassmannians: tau and Baker

§3.1: Infinite grassmannians

$n \geq 1$, $V = \mathbb{C}((z))^{\oplus n}$ and $V^+ = \mathbb{C}[[z]]^{\oplus n}$.

The infinite grassmannian of V , $\text{Gr}(V)$, is a scheme whose set of rational points is:

$$\left\{ \begin{array}{l} L \subset V \text{ such that the complex } L \rightarrow V/V^+ \\ \text{has finite dimensional kernel and cokernel} \end{array} \right\}$$

Further, if $\mathcal{L}$ denotes the universal submodule, the complex:

$$\mathcal{L} \xrightarrow{\delta} (V/V^+)_{\text{Gr}(V)}$$

is perfect. So, it defines the **determinant line bundle** Det on $\text{Gr}(V)$.

Moreover,

$$\det(\delta) \in H^0(\text{Gr}(V), \text{Det}^*)$$

Observe that $\det(\delta)_L = 0$ iff $L \cap V^+ \neq (0)$.

§3.2: tau functions

Call $\mathrm{Spf} \mathbf{C}[[t]]$ the “formal curve”, and define:

$$\widehat{C} := \mathrm{Spf} \mathbf{C}[[t_1]] \times \cdots \times \mathrm{Spf} \mathbf{C}[[t_n]]$$

Then $\Gamma := \varinjlim S^N \widehat{C}$ is the following group:

$$\Gamma(R) = \left\{ (g_1(z), \dots, g_n(z)) \text{ where } g_i(z) = 1 + \sum_{i>0} \frac{r_i}{z^i} \right. \\ \left. \text{with } r_i \in \mathrm{Rad} R \text{ and almost all vanish} \right\}$$

Componentwise multiplication gives an action:

$$\Gamma \times \mathrm{Gr}(V) \xrightarrow{\mu} \mathrm{Gr}(V)$$

Definition. *The tau function of $U \in \mathrm{Gr}^0(V)$ is:*

$$\tau_U := \mu^* \det(\delta)|_{\Gamma \times \{U\}}$$

Theorem. (Addition formula for tau) *The pullback of $\det(\delta)$ by: $\Pi^N \widehat{C} \times \{U\} \rightarrow \mathrm{Gr}(V)$ is:*

$$\frac{1}{\prod_{j=1}^n \Delta_j} \begin{vmatrix} f_1^1(t_{11}) & \cdots & f_1^1(t_{N1}) & \cdots & f_n^1(t_{1n}) & \cdots & f_n^1(t_{Nn}) \\ \vdots & & & & & & \vdots \\ f_1^M(t_{11}) & \cdots & f_1^M(t_{N1}) & \cdots & f_n^M(t_{1n}) & \cdots & f_n^M(t_{Nn}) \end{vmatrix}$$

where $M := N \cdot n$, $\{f^i = (f_1^i, \dots, f_n^i)\}_{1 \leq i \leq M}$ is a basis of $V^+ \cap z^N U$, and $\Delta_j := \prod_{i < k} (t_{ij} - t_{kj})$.

§3.3: Baker-Akhiezer functions

Since Γ is a group, consider the “subtraction” morphism:

$$\sigma : \hat{C} \times \Gamma \longrightarrow \Gamma$$

Definition. *The Baker-Akhiezer function of a point $U \in \text{Gr}^0(V)$ is defined by:*

$$\psi_U(z, t) := \exp \left(- \sum_{i>0} \frac{t_i}{z^i} \right) \frac{\sigma^* \tau_U(t)}{\tau_U(t)}$$

Theorem. *There exist polynomials $p_i(t)$ such that for every $U \in \text{Gr}^0(V)$ with $U \cap V^+ = (0)$:*

$$\psi_U(z, t) = z^{-1} \cdot \sum_{i>0} p_i(t) (f_1^i(z), \dots, f_n^i(z))$$

where $\{f^i = (f_1^i, \dots, f_n^i)\}_{i>0}$ is a basis of U .

Theorem. (Bilinear Identity) *Define the adjoint Baker-Akhiezer function: $\psi^*(z, t) := \psi_{U^\perp}(z, -t)$. It holds:*

$$\text{Res}_{z=\infty} \psi_U(z, t) \cdot \left(\psi_U^*(z, t') \right)^T dz = 0$$

§4: Multicomponent KP Hierarchy

§4.1: Pseudodifferential Operators (pdo)

A pdo is a formal expression of the type:

$$\sum_{-\infty < i \leq n} a_i \partial^i \quad n \in \mathbf{Z} \quad (1)$$

which is said to be of order n if $a_n \neq 0$.

For the case of the r -component KP hierarchy, one consider that the coefficients $M_{r \times r}(\mathbf{C})$ -valued functions and ∂ is a derivation.

The composition is defined by the “Leibnitz rule”:

$$\left(\sum_i a_i \partial^i \right) \left(\sum_j b_j \partial^j \right) := \sum_{i,j} \sum_{k \geq 0} \binom{i}{j} a_i (\partial^k b_j) \partial^{i+j-k}$$

For P as in (1) define:

$$P_- := \sum_{-\infty < i < 0} a_i \partial^i \quad \text{and} \quad P_+ := \sum_{0 \leq i \leq n} a_i \partial^i$$

§4.2: The hierarchy

Consider pdo:

$$L = \partial + L_1(s)\partial^{-1} + L_2(s)\partial^{-2} + \dots$$

$$C^{(i)} = E_i + C_1^{(i)}(s)\partial^{-1} + C_2^{(i)}(s)\partial^{-2} \dots$$

with coefficients in $M_{r \times r}(\mathbf{C}) \otimes \mathbf{C}[[\{s_{ij}\}_{i>0, 1 \leq j \leq r}]]$.
Let ∂_{ij} be $\frac{\partial}{\partial s_{ij}}$ and $\partial := \sum \partial_{1j}$.

The r -component KP hierarchy is the system:

$$\begin{cases} Lw = \bar{z} \cdot w \\ C^{(j)}w = w \cdot E_j \\ \partial_{ij}w = (L^i C^{(j)})_+ w \end{cases} \quad 1 \leq j \leq r, i > 0$$

for a formal function of the type:

$$w(\bar{z}, s) = (\text{Id} + \sum_{l < 0} w_l(s) \bar{z}^l) e^{\xi(s, \bar{z})}$$

If this is the case, w is called wave function.

Theorem. *There is a 1 – 1 correspondence between the set of solutions for the r -component KP hierarchy and the set of rational points $U \in \text{Gr}^0(\mathbf{C}((z))^{r^2})$ such that $U \cap \mathbf{C}[[z]]^{r^2} = (0)$.*

Remark. *The correspondence maps U to its Baker-Ahkieser function ψ_U .*

§5: Relation between tau and thetas: the Krichever map

Consider $\mathcal{U}(r, d)$ where $(r, d) = 1$ and a polarization Θ_F with $\text{rank } F = r$.

$$V := M_{r \times r}(\mathbf{C}) \otimes (\widehat{\mathcal{O}}_{C,p})_{(0)}$$

$$V^+ := M_{r \times r}(\mathbf{C}) \otimes \widehat{\mathcal{O}}_{C,p}$$

$$\widehat{\mathcal{U}}(r, d) := \left\{ (M, \beta) \text{ where } M \in \mathcal{U}(r, d) \right. \\ \left. \text{and } \beta : (M \otimes F)_p^\wedge \xrightarrow{\sim} V^+ \right\} \xrightarrow{\pi} \mathcal{U}(r, d)$$

The Krichever map:

$$K : \widehat{\mathcal{U}}(r, d) \hookrightarrow \text{Gr}^0(V) \\ (M, \beta) \mapsto H^0(C - p, M \otimes F)$$

Theorem.

$$K^* \text{Det} \xrightarrow{\sim} \pi^* \mathcal{O}(\Theta_F)$$

Proof. For a point $U = H^0(C - p, M \otimes F)$ the exact sequence:

$$0 \rightarrow U \cap V^+ \rightarrow U \xrightarrow{\delta} V/V^+ \rightarrow V/(U + V^+) \rightarrow 0$$

coincides with:

$$\begin{aligned} 0 \rightarrow H^0(C, M \otimes F) &\rightarrow H^0(C - p, M \otimes F) \rightarrow \\ &\rightarrow (\Sigma/\mathcal{O}_{C,p}) \otimes M_{r \times r}(\mathbb{C}) \rightarrow H^1(C, M \otimes F) \rightarrow 0 \end{aligned}$$

Theorem. Fix C , $p \in C$ and $(M, \beta) \in \hat{\mathcal{U}}(r, d)$. Then, there is a canonical morphism $\Gamma \rightarrow \mathcal{U}(r, d)$ and it holds that:

$$\theta_F|_{\Gamma} = \tau_U$$

where $U = H^0(C - p, M \otimes F)$.

Proof. $\mathcal{O}_C \rightarrow \hat{\mathcal{O}}_{C,p}$ induces $\hat{C} \rightarrow C^n$. Then:

$$\begin{array}{ccc} \hat{C} & \longrightarrow & C^n \\ \downarrow & & \downarrow \alpha_n \\ \Gamma & \longrightarrow & \mathcal{U}(r, d) \end{array} \quad \begin{array}{c} (x_1, \dots, x_n) \\ \downarrow \\ M(\sum(x_i - p)) \end{array}$$

This means that:

- non-abelian thetas give rise to solutions of the multicomponent KP hierarchy (§5); and,
- the addition formula for non-abelian theta functions (§2) is an instance of the addition formula for tau functions of points of the infinite grassmannian.