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Virasoro Algebra and Riemann Surfaces

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VNiVERSiDAD
D SALAMANCA

Definition

The Virasoro algebra, **Vir**, is the free Lie algebra generated by $\{L_n | n \in \mathbb{Z}\}$ and a central element c endowed with the bracket

$$[L_m, L_n] := (m - n)L_{m+n} + c \frac{m^3 - m}{12} \delta_{m, -n}$$

- the subalgebra $\mathbf{Vir}^+ := \langle \{L_n | n \geq -1\} \rangle$ is the positive half of **Vir**;
- **Vir** is the unique central extension of the Witt algebra,
 $\mathbf{Witt} := \mathbf{Vir} / \langle c \rangle$, up to isomorphisms.

FACTS

- **Witt** is isomorphic to the complexification of the Lie algebra of the orientation preserving diffeomorphisms group of S^1 , hence it acts on a given Riemann surface (and may change its conformal structure).
- In 2D quantum gravity, there exists a τ -function, τ , such that:

$$L_n \tau = 0 \text{ for } n \geq -1 \text{ and } \tau \text{ verifies the KP hierarchy}$$

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STUDY THE INTERPLAY BETWEEN:
 VIRASORO ALGEBRA, KP HIERARCHY AND RIEMANN SURFACES.

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Section's Outline

- 1 Motivation
 - Variational formulas
 - 2D Quantum Gravity and Intersection Theory



1st Motivation: Deformation of Riemann Surfaces

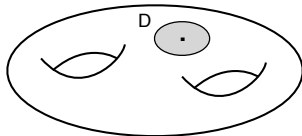
Fix a triple (X, p, z) : a Riemann Surface, a point and a local coordinate.

Set $D := \{q \in X \text{ s.t. } |z(q)| \leq \epsilon\}$ and an univalent function $g(z)$ on $D \setminus \{p\}$.

Consider the new surface

$$X^g := ((X - D^\circ) \sqcup D) / \sim$$

where $q \sim r$ when $q \in \partial(X - D^\circ)$, $r \in \partial D$ and $q = g(r)$.



Question

How does X change? (conformal structure, period matrix, theta function, etc.)

Follow classical approach of: Schiffer-Spencer, Rauch, Lebowitz, Kodaira-Spencer, etc.

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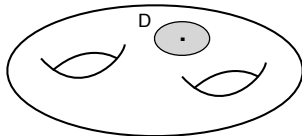
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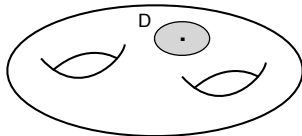
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Period Matrices

Fix cycles, a_i, b_i for $i = 1, \dots, g$, such that $a_i a_j = b_i b_j = 0$ and $a_i b_j = \delta_{i,j}$.
Fix 1-forms ω_i with $\int_{a_i} \omega_j = \delta_{i,j}$, the period matrix is $\Omega_{i,j}(X) := \int_{b_j} \omega_i$.

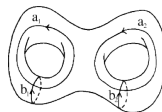
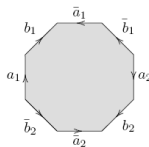


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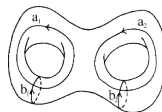
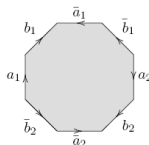
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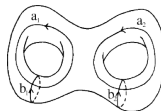
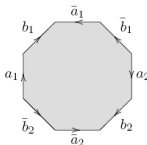
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And, at the first order, for $g(z) = z(1 + \epsilon g_0(z))$ (i.e. $\epsilon^2 = 0$)

$$\Omega_{i,j}(X^g) - \Omega_{i,j}(X) = \frac{-\epsilon}{2\pi\sqrt{-1}} \int_{\partial D} g_0(z) \partial_z f_i \omega_j$$

Witt action

Recall that the complexification of the Lie algebra of orientation preserving diffeomorphisms of $S^1 \simeq \partial D$ is isomorphic to the Lie algebra of vector fields on S^1 and, explicitly

$$g(z) = z(1 + \epsilon z^n) \quad \longleftrightarrow \quad L_n := -z^{n+1} \partial_z \in \mathbb{C}((z)) \partial_z$$

and note that $[L_m, L_n] = (m - n)L_{m+n}$; i.e. it is isomorphic to **Witt**.

Let $\widehat{\mathcal{M}}$ be the moduli space of triples (X, p, z) , then the above computations yield the Kodaira-Spencer map / Virasoro Uniformization

$$0 \rightarrow H^0(X - p, \mathbb{T}_X) \rightarrow \mathbf{Witt} = \mathbb{C}((z)) \partial_z \rightarrow T_{(X, p, z)} \widehat{\mathcal{M}} = \varprojlim_n H^1(X, \mathbb{T}_X(-np)) \rightarrow 0$$

Properties:

- L_n changes the formal coordinate z for $n \geq 0$,
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Tau and Theta: KP hierarchy

Given (X, p, z) with period matrix Ω_X .

Take 1-forms, η_n , holomorphic at $X - p$ and $\eta_n = d(z^{-n}) + O(1)$ at p .

Consider the quadratic form $Q(t) = \sum_{n,m \geq 1} q_{nm} t_n t_m$ where

$$\int^x \eta_n = z^{-n} - 2 \sum_{m \geq 1} q_{nm} \frac{z^m}{m}$$

and the $g \times \infty$ -matrix A whose j -th row is the expansion of ω_j at p .

τ -function

$$\tau_X(t) := \exp Q(t) \Theta(A(t); \Omega_X) \quad t = (t_1, t_2, \dots)$$

Remarks:

- The previous computations and the Heat equation allows us to express the action of L_n on τ as a second order differential operator in $t_1, t_2, \dots$
- The τ -function associated to (X, p, z) is a solution of the KP hierarchy.
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Consider the quadratic form $Q(t) = \sum_{n,m \geq 1} q_{nm} t_n t_m$ where

$$\int^x \eta_n = z^{-n} - 2 \sum_{m \geq 1} q_{nm} \frac{z^m}{m}$$

and the $g \times \infty$ -matrix A whose j -th row is the expansion of ω_j at p .

τ -function

$$\tau_X(t) := \exp Q(t) \Theta(A(t); \Omega_X) \quad t = (t_1, t_2, \dots)$$

Remarks:

- The previous computations and the Heat equation allows us to express the action of L_n on τ as a second order differential operator in $t_1, t_2, \dots$
- The τ -function associated to (X, p, z) is a solution of the KP hierarchy.
- One can add a line bundle on X to the previous construction.



2nd Motivation: Intersection Theory on Moduli

Let $\bar{\mathcal{M}}_{g,n}$ be the compactification of the moduli stack of genus g Riemann Surfaces with n punctures.

Let L_i be the line bundle on $\bar{\mathcal{M}}_{g,n}$ whose fiber at $(X, x_1, \dots, x_n)$ is the cotangent space of X at x_i .

Let $c_1(L_i)$ its Chern class (relevant for studying the cohomology ring).

Let $\langle \tau_0^{k_0} \tau_1^{k_1} \dots \rangle := \int_{\bar{\mathcal{M}}_{g,n}} \prod c_1(L_i)^{k_i}$ be the intersection index of $\prod c_1(L_i)^{k_i}$.

Witten's generating function

$$F(t_0, t_1, \dots) := \sum \langle \tau_0^{k_0} \tau_1^{k_1} \dots \rangle \prod_{i \geq 0} \frac{t_i^{k_i}}{k_i!}$$

- F is the partition function in the standard matrix model theory.
- Generalizations by [Eguchi-Hori-Xiong, Dubrovin-Zhang].



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KdV and Virasoro

Theorem [Witten-Kontsevich]

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- $\exp(F)$ in $T_{2i+1} := \frac{t_i}{(2i+1)!!}$ is a τ -function for the KdV hierarchy
- F satisfies the Virasoro constraints $L_n F = 0$ for $n \geq -1$.

[Witten, Kontsevich, Dijkgraaf-Verlinde-Verlinde, Kac-Schwartz, Givental, etc.]

- $n = -1$ the string equation,
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Section's Outline

- 2 Grassmannian, Actions and Differential Operators
 - Grassmannian and Fock Space
 - Actions on Grassmannians
 - Translating the Problem



Sato Grassmannian and KP

We know that $\text{Gr} := \text{Gr}(\mathbb{C}((z)))$ is the space of τ -functions of the KP hierarchy ([Segal-Wilson, Sato-Sato, Krichever, etc.])

$$\left\{ \begin{array}{l} \tau(t) \in \mathbb{C}[[t_1, t_2, \dots]] \\ \text{satisfying KP hierarchy} \end{array} \right\} \hookrightarrow \text{Gr} := \left\{ \begin{array}{l} U \subset \mathbb{C}((z)) \text{ s.t. } \pi_U : U \rightarrow \mathbb{C}((z))/\mathbb{C}[[z]] \\ \text{has finite dimensional ker and coker} \end{array} \right\}$$

The image consists of those U s.t. $U \oplus \mathbb{C}[[z]] \simeq \mathbb{C}((z))$. For such U

$$U \in \text{Gr} \rightsquigarrow \wedge \ker \pi_U \otimes \wedge (\text{coker } \pi_U)^* \subset \wedge^{\frac{\infty}{2}} \mathbb{C}((z)) \rightsquigarrow \tau_U(t) \in \mathbb{C}[[t_1, t_2, \dots]]$$

where the last identification is the Boson-Fermion correspondence or, explicitly,

$$\tau_U(t) := \frac{1}{\det(U \rightarrow \mathbb{C}((z))/\mathbb{C}[[z]])} \det(\exp(\sum t_i z^{-i}) U \rightarrow \mathbb{C}((z))/\mathbb{C}[[z]])$$

Every τ -function with $\tau(0) = 1$ arises in this way.

There is a link between geometric objects and the KP hierarchy. Indeed,

$$\widehat{M} \xrightarrow{\text{Krichever}} \text{Gr } \mathbb{C}((z)) \xrightarrow{\text{Plücker}} \mathbb{P}\mathbb{C}[[t_1, t_2, \dots]]$$

$$(X, p, z) \mapsto U := H^0(X - p, \mathcal{O}) \mapsto \tau_U(t) = \exp Q(t) \Theta(A(t); \Omega_X)$$

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Action of groups

Any subgroup G of the linear group of $\mathbb{C}((z))$ acts on the Grassmannian

$$\begin{aligned} G \times \text{Gr} &\longrightarrow \text{Gr} \\ (g, U) &\longmapsto g(U) \end{aligned}$$

Let Det be the determinant line bundle on Gr ; whose fiber at U is $\wedge \ker(\pi_U) \otimes (\wedge \text{coker } \pi_U)^*$. Then

$$g^* \text{Det} \simeq \text{Det} \quad \forall g \in G$$

Thus, there is a central extension

$$0 \rightarrow \mathbb{C}^* \rightarrow \tilde{G} := \left\{ (\tilde{g}, g) \text{ s.t. } \begin{array}{ccc} \text{Det} & \xrightarrow{\tilde{g}} & \text{Det} \\ \downarrow & & \downarrow \\ \text{Gr} & \xrightarrow[g]{\sim} & \text{Gr} \end{array} \right\} \rightarrow G \rightarrow 0$$

$$\rho: G \curvearrowright \text{Gr} \quad \Longrightarrow \quad \tilde{\rho}: \tilde{G} \curvearrowright H^0(\text{Gr}, \text{Det}^*) \simeq \mathbb{C}[[t_1, t_2, \dots]]$$

and $\tilde{G}$ is trivial iff the 2-cocycle of the central extension is trivial.



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For $U \in \text{Gr}$, at the level of tangent spaces,

$$\text{Lie } G \rightarrow T_U \text{Gr} = \text{Hom}(U, \mathbb{C}((z))/U)$$

$$L \mapsto U \hookrightarrow \mathbb{C}((z)) \xrightarrow{\rho(L)} \mathbb{C}((z)) \rightarrow \mathbb{C}((z))/U$$

and the above construction yields

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Remark. For $G = \text{Aut } \mathbb{C}((z))$, it holds that $\text{Lie } G = \mathbf{Witt}$ and $\text{Lie } \tilde{G} = \mathbf{Vir}$.

For the case of the Lie algebra of differential operators, the cocycle is the Gelfand-Fuchs cocycle

$$\langle a(z)\partial_z^r, b(z)\partial_z^s \rangle = \frac{r!s!}{(r+s+1)!} \text{Res}_{z=0} (\partial_z^{s+1} a(z) \cdot \partial_z^r b(z)) dz$$

Theorem

Let ρ be a representation of $\mathbf{Vir}^+$, then the cocycle is trivial and

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Translating the Problem

Now, we can go back and forth from Gr to the Boson Fock Space:

Action on $\mathbb{C}((z))$	Action on $\mathbb{C}[[t_1, t_2, \dots]]$
z^n with $n \geq 1$	nt_n
z^{-n} with $n \geq 1$	∂_{t_n}
1	0
$z^n(z\partial_z + \frac{n+1}{2})$ with $n \geq 1$	$\frac{1}{2} \sum_{i=1}^{n-1} i(n-i)t_i t_{n-i} + \sum_{i=1}^{\infty} (n+i)t_{n+i}\partial_{t_i}$
$z^{-n}(z\partial_z + \frac{1-n}{2})$ with $n \geq 1$	$\frac{1}{2} \sum_{i=1}^{n-1} \partial_{t_i} \partial_{t_{n-i}} + \sum_{i=1}^{\infty} it_i \partial_{t_{n+i}}$
For $U \subset \mathbb{C}((z))$	For $\tau_U(t) \in \mathbb{C}[[t_1, t_2, \dots]]$
$U \in \text{Gr}$	KP hierarchy
$z^{-2}U \subseteq U$	$\partial_{t_{2i}}\tau_U(t) = 0$ for all i (KdV, provided KP)
$\rho(L)(U) \subseteq U$	$\tilde{\rho}(L)\tau_U(t) = 0$

This is ok for one element, but

For a group G acting on $\mathbb{C}((z))$, one has to check that the extension $\tilde{G}$ splits.



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Section's Outline

- 3 Construction of solutions
 - Representations of $\mathbf{Vir}^+$
 - Invariant Subspaces
 - Solutions



Representations of Virasoro

Theorem (P.)

There exists a bijection

$$\left\{ \begin{array}{l} \rho : \mathbf{Vir}^+ \rightarrow \text{Diff}^1(\mathbb{C}((z))) \\ \text{of Lie alg. such that } \rho \neq 0 \end{array} \right\} \xleftrightarrow{1-1} \left\{ \begin{array}{l} \text{triples } (h(z), c, b(z)) \text{ such that} \\ h'(z) \neq 0, c \in \mathbb{C}, b(z) \in \mathbb{C}((z)) \end{array} \right\}$$

which is explicitly given by:

$$\rho(L_i) = \frac{-h(z)^{i+1}}{h'(z)} \partial_z - (i+1)c \cdot h(z)^i + \frac{h(z)^{i+1}}{h'(z)} b(z)$$

Theorem (P.)

Let $U \subseteq \mathbb{C}((z))$. If $\rho(L_i)(U) \subseteq U$ for all $i \geq -1$, then

$$A_U := \{f \in \mathbb{C}((z)) \text{ such that } f \cdot U \subseteq U\} = \mathbb{C}[h]$$

$$A_U^1 := \{D \in \text{Diff}^1 \mathbb{C}((z)) \text{ such that } D(U) \subseteq U\} = \mathbb{C}[h] \oplus \text{Im } \rho$$

If $h(z) = z^{-2}$, $\text{Res}_{z=0} b(z) = \frac{3}{2}$ and $\dots$, then the central extension $\widetilde{A}_U^1$ splits.



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Auxiliary Functions

From now on $h(z) = z^{-2}$.

Let S be the Schwarzian derivative $S(h(z)) := \frac{h'''(z)}{h'(z)} - \frac{3}{2} \left(\frac{h''(z)}{h'(z)} \right)^2$

The Airy equation is

$$w''(z) + \frac{1}{2} S(h(z)) w(z) = 0 \quad (1)$$

If $w(z)$ is a solution, then $\frac{w'(z)}{w(z)}$ is meromorphic.

Let ρ be a representation, corresponding to a triple $(h(z), c, b(z))$, such that there is a formal element $w(z)$ and a polynomial $u(h(z))$ with

$$\frac{w'(z)}{w(z)} = b(z) - u(h(z))h'(z) + \frac{1}{2}h''(z) \in \mathbb{C}[h(z)] + z^{-1}\mathbb{C}[[z]] \quad (2)$$

Set $v(z)$ such that $\frac{v'(z)}{v(z)}$ makes sense as an element of $\mathbb{C}((z))$.

Given ρ and $v(z)$, we consider a induced representation on the $\mathbb{C}((z))$ -vector space generated by $v(z)$; i.e. $\mathbb{C}((z)) \otimes v(z)$, by

$$\rho^v(L)(f(z) \otimes v(z)) := (\rho(L)(f(z)) + \sigma(\rho(L))f(z) \frac{v'(z)}{v(z)}) \otimes v(z)$$

where σ is the symbol of the differential operator.



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Invariant Subspaces

Theorem (Existence of solutions, P.)

Let ρ be given. Consider solutions $w_1(z)$ of (1) and $w_2(z)$ of (2) and $v(z) := w_1(z)w_2(z)$. The subspace

$$\mathcal{U} := \mathbb{C}[h(z)] \otimes \langle 1 \otimes v(z), \rho^v(L_{-1})(1) \otimes v(z) \rangle \subset \mathbb{C}((z)) \otimes v(z)$$

fulfills

- ① $\mathcal{U}$ belongs to the Sato Grassmannian of $\mathbb{C}((z)) \otimes v(z)$;
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Idea of proof:

- ① It follows from the choice of $w_2(z)$ as in equation (2).
- ② By the very construction $z^{-2} \in \mathbb{C}[h(z)] \subseteq A_{\mathcal{U}}$.
- ③ For $\rho^v(L_i)$ consider:

$$\rho^v(L_i)w_1(z)w_2(z) = \rho^v(L_i)w_1(z)w_2(z) + \rho^v(L_i)w_1(z)w_2(z) = \rho^v(L_i)w_1(z)w_2(z) + \rho^v(L_i)w_1(z)w_2(z)$$

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Solutions

Proposition

Given ρ , let $\mathcal{U}_1, \mathcal{U}_2$ be two subspaces corresponding to two different solutions of the Airy equation. Then, they differ in the exponential of a linear function; i.e.,

$$\partial_{t_{2i+1}} \partial_{t_{2j+1}} \log \tau_{\mathcal{U}_1} = \partial_{t_{2i+1}} \partial_{t_{2j+1}} \log \tau_{\mathcal{U}_2} \quad \forall i, j, \geq -1$$

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- If $w_1(z)$ and $w_2(z)$ are two solutions of the Airy equation (1), then

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- The corresponding stable subspaces are related by

$$\mathcal{U}_1 = \frac{\alpha h(z) + \beta}{\gamma h(z) + \delta} \cdot \mathcal{U}_2$$

It is an identity of $\mathbf{Vir}^+$ -modules.

- If $\frac{\alpha h(z) + \beta}{\gamma h(z) + \delta}$ can be expanded as $g(h(z)^{-1}) \in \mathbb{C}((h(z)^{-1}))$, then

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It is an identity of $\mathbf{Vir}^+$ -modules.

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Solutions

Proposition

Given ρ , let $\mathcal{U}_1, \mathcal{U}_2$ be two subspaces corresponding to two different solutions of the Airy equation. Then, they differ in the exponential of a linear function; i.e.,

$$\partial_{t_{2i+1}} \partial_{t_{2j+1}} \log \tau_{\mathcal{U}_1} = \partial_{t_{2i+1}} \partial_{t_{2j+1}} \log \tau_{\mathcal{U}_2} \quad \forall i, j, \geq -1$$

Idea of proof:

- If $w_1(z)$ and $w_2(z)$ are two solutions of the Airy equation (1), then

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Stating the properties of $\mathcal{U}$ in terms of its τ -function, one obtains that

- $\tau_{\mathcal{U}}(t)$ satisfies the KdV hierarchy
- $\tau_{\mathcal{U}}(t)$ satisfies the Virasoro constraints $\rho^{\vee}(L_i)\tau_{\mathcal{U}}(t) = 0$ for $i \geq -1$.

Theorem (Uniqueness, P.)

The Virasoro constraints admit a unique solution (provided KdV and up to the exponential of a linear function).

A function can solve only one set of Virasoro constraints at most.

In other words, let (ρ_j, τ_j) be given such that $\rho_j(L_i)\tau_j = 0$ for $j = 1, 2$.

The following conditions are equivalent

- $\partial_{t_{2j+1}} \partial_{t_{3j+1}} \log \tau_1 = \partial_{t_{2j+1}} \partial_{t_{3j+1}} \log \tau_2$
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Section's Outline

- 4 Further Applications
 - Geometric interpretation of the construction
 - Universal Family of reps of $\mathbf{Vir}^+$
 - Representations of Virasoro and $\mathfrak{sl}(2)$



Geometric interpretation of the construction

Theorem (P.)

Given ρ with $h(z) = z^{-2}$.

We associate to ρ a pair $(\mathcal{E}, \nabla)$, consisting of a vector bundle of rank 2 over the projective line, and a connection on it.

The proof relies on the Krichever construction. Let $\mathcal{U}$ be an invariant subspace (with the previous notations).

- $\mathbb{P}_1$ is obtained by gluing $\text{Spec } A_{\mathcal{U}} = \text{Spec } \mathbb{C}[h(z)]$ with the valuation given by $h(z)^{-1}$.
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$$\begin{aligned} \mathcal{U} \otimes_{\mathbb{C}} \mathbb{C}[h(z)] &\rightarrow \mathcal{U} \otimes_{\mathbb{C}} \Omega_{\mathbb{C}[h(z)]/\mathbb{C}} \\ f \otimes a &\mapsto \rho(L_{-1})f \otimes a \, dh - f \otimes da \end{aligned}$$

(because of $\rho(L_{-1})\mathcal{U} \subseteq \mathcal{U}$ and the properties of ρ)

Remark Given $(\mathbb{P}_1, \rho, \bar{z}, \mathcal{E}, \nabla)$, note $\text{Der}(\mathcal{O}_{\rho})_{(0)}^{\wedge} \simeq \mathbf{Witt} = \langle \{\bar{z}^{n+1}\partial_{\bar{z}}\} \rangle$, and

$$\bar{z}^{n+1}\partial_{\bar{z}} \longmapsto \nabla_{\bar{z}^{n+1}\partial_{\bar{z}}} \quad \text{i.e.} \quad \nabla_{\bar{z}^{n+1}\partial_{\bar{z}}}(f) := \langle \nabla(f), \bar{z}^{n+1}\partial_{\bar{z}} \rangle$$

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Universal Family of reps of $\mathbf{Vir}^+$

Let $\mathbf{m} := (m_1, m_2, \dots)$ be non-negative integers with $m_i = 0, \forall i \gg 0$.

Let $\mathbf{s} := (s_1, s_2, \dots)$ be variables. Define

$$|\mathbf{m}| := \sum_{i \geq 1} im_i \quad \|\mathbf{m}\| := \sum_{i \geq 1} m_i \quad \mathbf{m}! := \prod_{i \geq 1} m_i! \quad \mathbf{s}^{\mathbf{m}} := \prod_{i \geq 1} s_i^{m_i}$$

$$\begin{aligned} \bar{L}'_n(\mathbf{s}) := & -\frac{1}{2} \sum_{\mathbf{m}} \frac{(-1)^{\|\mathbf{m}\|}}{\mathbf{m}!(2|\mathbf{m}|+1)!!} \mathbf{s}^{\mathbf{m}} \partial_{q_{|\mathbf{m}|+n+1}} + \sum_{i=0}^{\infty} (i + \frac{1}{2}) q_i \partial_{q_{i+n}} + \\ & + \frac{1}{2} \sum_{i=1}^n \partial_{q_{i-1}} \partial_{q_{n-i}} + \frac{q_0^2}{4} \delta_{n,-1} + \frac{1}{16} \delta_{n,0} \end{aligned}$$

[Dijkgraaf-Verlinde-Verlinde, Witten-Kontsevich, Givental, Mulase-Safnuk, Liu-Xu] are of this form. They fulfill $[\bar{L}'_i(\mathbf{s}), \bar{L}'_j(\mathbf{s})] = (i-j)\bar{L}'_{i+j}(\mathbf{s})$ for $i, j \geq -1$.

Theorem (P.)

$\rho'_s(L_i) := \bar{L}'_i(\mathbf{s})$ is attached to the data $(h(z) = z^{-2}, c = \frac{1}{2}, b_s(z))$ where:

$$b_s(z) := -\sum_{\mathbf{m}} \frac{(-1)^{\|\mathbf{m}\|}}{\mathbf{m}!(2|\mathbf{m}|+1)!!} \mathbf{s}^{\mathbf{m}} z^{-2|\mathbf{m}|-4} - \frac{3}{2} z^{-1}$$



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Representations of Virasoro and $\mathfrak{sl}(2)$

A representation $\rho : \mathbf{Vir}^+ \rightarrow \text{Diff}^1 \mathbb{C}((z))$ is associated to $(h(z), c, b(z))$

$$\rho(L_i) = \frac{-h(z)^{i+1}}{h'(z)} \partial_z - (i+1)c \cdot h(z)^i + \frac{h(z)^{i+1}}{h'(z)} b(z)$$

and, from the explicit expression, it is determined by $\{\rho(L_{-1}), \rho(L_0), \rho(L_1)\}$.

Set a Chevalley basis $\{e, f, h\}$ of $\mathfrak{sl}(2)$, $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$.

Note $\mathfrak{sl}(2) \simeq \langle L_{-1}, L_0, L_1 \rangle \subset \mathbf{Vir}^+$ by sending

$$f \mapsto L_{-1} \quad h \mapsto -2L_0 \quad e \mapsto -L_1$$

Question

How general is this fact? That is, when ρ is determined by $\rho|_{\mathfrak{sl}(2)}$?
 It holds true for infinite dimensional simple weight representations.

An $\mathfrak{sl}(2)$ -module V is called weight if $V = \bigoplus_{\lambda \in \mathbb{C}} \ker(h - \lambda)$.

Similarly, an $\mathbf{Vir}^+$ -module V is called weight if $V = \bigoplus_{\lambda \in \mathbb{C}} \ker(L_0 - \lambda)$.



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Question

How general is this fact? That is, when ρ is determined by $\rho|_{\mathfrak{sl}(2)}$?
 It holds true for infinite dimensional simple weight representations.

An $\mathfrak{sl}(2)$ -module V is called weight if $V = \bigoplus_{\lambda \in \mathbb{C}} \ker(h - \lambda)$.

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Representations of Virasoro and $\mathfrak{sl}(2)$

A representation $\rho : \mathbf{Vir}^+ \rightarrow \text{Diff}^1 \mathbb{C}((z))$ is associated to $(h(z), c, b(z))$

$$\rho(L_i) = \frac{-h(z)^{i+1}}{h'(z)} \partial_z - (i+1)c \cdot h(z)^i + \frac{h(z)^{i+1}}{h'(z)} b(z)$$

and, from the explicit expression, it is determined by $\{\rho(L_{-1}), \rho(L_0), \rho(L_1)\}$.

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Simple weight modules

There are three types of infinite dimensional simple weight $\mathfrak{sl}(2)$ -modules.

- *highest weight module* $M(\lambda)$ for $\lambda \in \mathbb{C} \setminus \{0, 1, 2, \dots\}$.
- *lowest weight module* $\tilde{M}(\lambda)$. Let $\lambda \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$.
- *dense modules* $V(\xi, \tau)$. For $\xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$ with $\tau \neq (\mu + 1)^2$.

$V(\xi, \tau)$ is the vector space generated by $\{v_\mu | \mu \in \xi\}$ endowed the action

$$f(v_\mu) = v_{\mu-2}$$

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- *highest weight modules*.
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$$L_i \cdot v_n := (a \cdot i + b - n)v_{n+i}$$



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$$f(w_i) = w_{i+1}$$

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$$e(w_i) = \begin{cases} i(\lambda - i + 1)w_{i-1} & \text{for } i \neq 0 \\ 0, & \text{for } i = 0 \end{cases}$$

$\{\lambda, \lambda - 2, \lambda - 4, \dots\}$ is the set of weights. We say that λ is the highest.

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Every infinite dimensional simple weight $\mathfrak{sl}(2)$ -module admits a compatible structure of $\mathbf{Vir}^+$ -module.

- We obtain explicit expressions for these actions.
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Remarks:

- ① Let M be an $\mathfrak{sl}(2)$ -module and σ the action. Let $T \in \text{End}(M)$. Define ρ by

$$\rho|_{\mathfrak{sl}(2)} = \sigma \quad \text{and} \quad \rho(L_{2+i}) := \frac{1}{i!} \text{ad}(\sigma(e))^i(T) \quad i \geq 0$$

Then, ρ defines a compatible $\mathbf{Vir}^+$ action, if and only if

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Thanks for your attention!

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