

Covers of curves and the Schottky problem for curves with automorphisms

Workshop “Local fields, algebraic geometry, generalized KP-hierarchy”, Berlin, June, 2-7th, 2005

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based on:

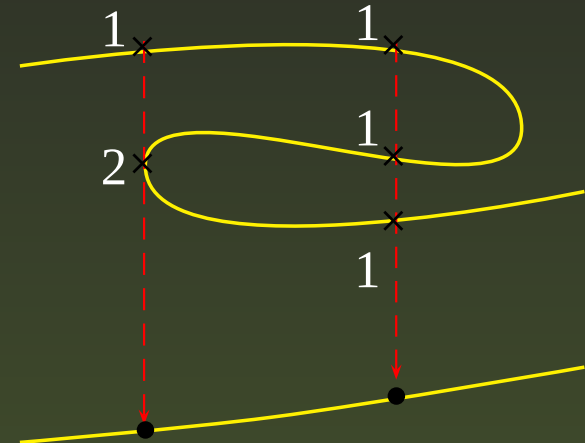
- joint work with J. Muñoz Porras (<http://arxiv.org/abs/math/0207091>)
- joint work with E. Gómez, J. Muñoz Porras, S. Recillas and R. Rodríguez

Problem:

- Give conditions (on the theta function) for a curve Y to be a cover of another curve X with prescribed ramification type.

Application:

- **Schottky problem for R.S. with automorphisms:** Give conditions that a theta function of a p.p.a.v. should satisfy in order to be a jacobian theta function of a Riemann surface with a non-trivial automorphism.



Introduction

Theorem. *Let X be an irreducible p.p.a.v. of dimension $g > 1$. Then the following conditions are equivalent.*

1. $\exists C$, genus g projective irreducible smooth curve, with an order p automorphism $\sigma_C : C \rightarrow C$ such that $X \simeq J(C)$ as a p.p.a.v.
2. *There exist $g \times \infty$ -matrices $A^{(1)}, \dots, A^{(p)}$ of rank g and symmetric quadratic forms $Q^{(1)}, \dots, Q^{(p)}$, such that*
 - a) $\exists \xi_0 \in \mathbb{C}^g$, the corresponding BA-functions satisfy the $(1, \dots, 1)$ -KP hierarchy

$$\text{Res} \left(\sum_{j=1}^p z^{-\delta_{ju} - \delta_{jv}} \psi_{u, \xi_0}^{(j)}(z, t) \psi_{v, \xi_0}^{*(j)}(z, s) \right) dz = 0$$

- b) $\exists \xi_1 \in \mathbb{C}^g$ (depending on ξ_0) such that

$$\text{Res} \left(\sum_{j=1}^p z^{-\delta_{ju} - \delta_{jv}} \psi_{v+1, \xi_0}^{(j+1)}(z, \sigma^*(t)) \psi_{u, \xi_1}^{*(j)}(z, s) \right) dz = 0$$

where $\sigma^*(t) := (t^{(p)}, t^{(1)}, t^{(2)}, \dots, t^{(p-1)})$.

1. Preliminaries on Grassmannians, formal groups, τ -functions and BA-functions.
2. Characterization of covers and bilinear identities.
3. Geometric meaning of Γ_V and τ -function in the Jacobian case.
4. Characterization of Jacobians of curves with automorphisms as finite dimensional orbits.
5. A solution for the Schottky problem for curves with automorphisms.
6. Open questions.

1.1. Preliminaries

Consider the $\mathbb{C}((z))$ -algebra

$$V = \mathbb{C}((z_1)) \times \cdots \times \mathbb{C}((z_p))$$

where $z = (z_1, \dots, z_p)$ and subspaces $V^+ = \mathbb{C}[[z_1]] \times \cdots \times \mathbb{C}[[z_p]]$ and $V^- = z_1^{-1}\mathbb{C}[z_1^{-1}] \times \cdots \times z_p^{-1}\mathbb{C}[z_p^{-1}]$.

Let Γ_V be the formal group scheme

$$\Gamma_V(R) := \left\{ \begin{array}{l} \text{connected component} \\ \text{of 1 in } (V \hat{\otimes}_{\mathbb{C}} R)^* \end{array} \right\} = \left\{ \begin{array}{l} a_{-n}z^{-n} + \cdots + a_0 + a_1z + \cdots \\ \text{where } a_{-i} \in \text{Rad}(R) \text{ and } a_0 \in R^* \end{array} \right\}$$

Then, $\Gamma_V = \Gamma_V^- \times \bar{\Gamma}_V^+$ where $\Gamma_V^+ := \{a_0 + a_1z + \cdots\}$ and $\Gamma_V^- := \{a_{-i}z^{-i} + \cdots + a_{-1}z^{-1} + 1\}$.

There is a natural morphism, the Abel map,

$$\hat{C}_V := \text{Spf} \left(\mathbb{C}[[\bar{z}_1]] \times \cdots \times \mathbb{C}[[\bar{z}_p]] \right) \xrightarrow{\phi_V} \Gamma_V^-$$

such that (Γ_V^-, ϕ_V) is the Albanese variety of $\hat{C}_V$.

1.2. Preliminaries

The **infinite Grassmannian** scheme

$$\mathrm{Gr}(V) := \left\{ U \subset V \text{ such that } U \rightarrow V/V^+ \text{ has } \right. \\ \left. \text{finite dimensional ker and coker} \right\}$$

On the connected component where $\ker$ and Coker have the same dimension, one has the determinant bundle

$$\mathrm{Det} := \mathrm{Det} \left(U \xrightarrow{\delta} V/V^+ \right)$$

and a global section $\det(\delta) \in H^0(\mathrm{Gr}(V), \mathrm{Det}^*)$.

Then, the **τ -function** of $U \in \mathrm{Gr}(V)$ is a function on Γ_V^- given by

$$\tau_U(t_1, t_2, \dots) := \text{normalization of } \det(\delta) \left(\left(\sum_i \frac{t_i}{z^i} \right) U \right)$$

1.3. Preliminaries

The u -th Baker-Akhiezer function of a point $U \in \text{Gr}(V)$ is the V -valued function $\psi_{u,U}(z, t)$

$$\left(\exp \left(- \sum_{i \geq 1} \frac{t_i^{(1)}}{z_1^i} \right) \frac{\tau_{U_{u1}}(t + [z_1])}{\tau_U(t)}, \dots, \exp \left(- \sum_{i \geq 1} \frac{t_i^{(p)}}{z_p^i} \right) \frac{\tau_{U_{up}}(t + [z_p])}{\tau_U(t)} \right)$$

where $1 \leq u \leq r$, $U_{uv} := (1, \dots, z_u, \dots, z_v^{-1}, \dots, 1) \cdot U$ and $t + [z_v] := (t^{(1)}, \dots, t^{(v)} + [z_v], \dots, t^{(r)})$.

Theorem. Let $U \in \text{Gr}^0(V)$. Then:

$$\psi_{u,U}(z, t) = (1, \dots, z_u, \dots, 1) \cdot \sum_{i > 0} (\psi_{u,U}^{(i,1)}(z_1), \dots, \psi_{u,U}^{(i,r)}(z_r)) p_{ui,U}(t)$$

where:

$$\{ (\psi_{u,U}^{(i,1)}(z_1), \dots, \psi_{u,U}^{(i,r)}(z_r)) \mid i > 0, 1 \leq u \leq r \}$$

is a basis of U and $p_{ui,U}(t)$ are functions in t .

1.4. Preliminaries

Theorem. *A function as above is a BA function of a point of $\text{Gr}(V)$ if and only if it satisfies the $(1, \dots, 1)$ -KP-hierarchy*

$$\text{Res}_{z=0} \text{Tr} \left(\frac{\psi_{u,U}(z, t)}{(1, \dots, z_u, \dots, 1)} \cdot \frac{\psi_{v,U}^*(z, s)}{(1, \dots, z_v, \dots, 1)} \right) dz = 0$$

for all $u, v \in \{1, \dots, p\}$.

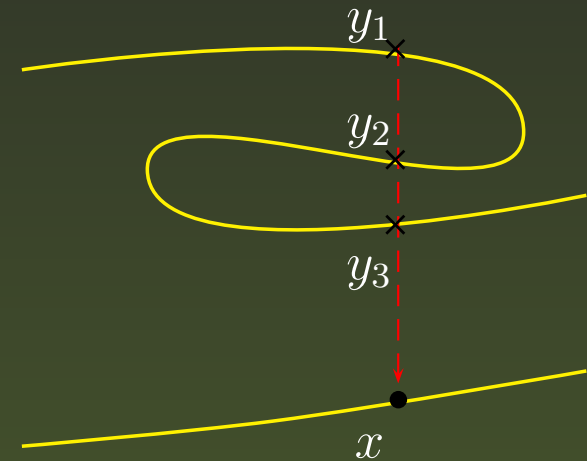
Equivalently, in terms of p.d.e.'s for the τ -function

$$\sum_{j=1}^p \sum p_{\alpha_1}(-t^{(j)}) p_{\beta_1}(\tilde{\partial}_{t^{(j)}}) \tau_{U_{u,j}}(t) \cdot p_{\alpha_2}(s^{(j)}) p_{\beta_2}(-\tilde{\partial}_{s^{(j)}}) \tau_{U_{jv}}(s) = 0$$

2.1. Covers of curves

Let us consider data $(X, Y, \pi, x, \bar{y}, t_x, t_{\bar{y}})$ where:

- $\pi : Y \rightarrow X$ is a cover of curves.
- $x \in X$ is a smooth point.
- $\pi^{-1}(x) = \bar{y} = \{y_1, \dots, y_p\} \subset Y$ are pairwise distinct smooth points.
- $t_x : \hat{\mathcal{O}}_{X,x} \simeq \mathbb{C}[[z]]$
- $t_{\bar{y}} : \hat{\mathcal{O}}_{Y,\bar{y}} \simeq \mathbb{C}[[z_1]] \times \dots \times \mathbb{C}[[z_p]]$



Consider the Krichever map:

$$\{ \text{geometric data} \} \longrightarrow \text{Gr}(V)$$

$$(Y, X, \pi, x, \bar{y}, t_x, t_{\bar{y}}) \longmapsto t_{\bar{y}}(H^0(Y - \bar{y}, \mathcal{O}_Y))$$

2.2. Covers of curves

Theorem. *A point $U \in \text{Gr}(V)$ comes from geometric data (via the Krichever morphism) if and only if*

$$U \cdot U = U$$

$$\text{Tr}(U) \subseteq U$$

where $\text{Tr} : V \rightarrow \mathbb{C}((z))$ is the trace map of V as a finite $\mathbb{C}((z))$ -algebra.

It follows that the image of the Krichever map is a closed subscheme of the infinite grassmannian.

Further, a point U comes from geometric data if and only if

$$\left(\sum_{l=1}^r \sum_{j=1}^{k_l} \sum_{i=1}^{e_j^{(l)}} \frac{\psi_U^{(l,j)}(\xi_{e_j^{(l)}}^i z_l^{1/e_j^{(l)}}, t)}{(\xi_{e_j^{(l)}}^i z_l^{1/e_j^{(l)}})^{\delta_{al}\delta_{bj}-u_{l,j}}} \right) \cdot \left(\sum_{l=1}^r \sum_{j=1}^{k_l} \sum_{i=1}^{e_j^{(l)}} \frac{\psi_U^{(l,j)*}(\xi_{e_j^{(l)}}^i z_l^{1/e_j^{(l)}}, t)}{(\xi_{e_j^{(l)}}^i z_l^{1/e_j^{(l)}})^{u_{l,j}-1+\delta_{cl}\delta_{dj}}} \right) \frac{dz}{z}$$

has residue equal to 0 in $z = 0$ (and a similar equation for the condition $U \cdot U = U$).

2.3. Covers of curves

Proof.

- $U \cdot U = U$ implies that U is a subalgebra of V and, therefore, $\exists(Y, \bar{y}, t_{\bar{y}})$ such that $U = t_{\bar{y}}(H^0(Y - \bar{y}, \mathcal{O}_Y))$.
- One has a commutative diagram

$$\begin{array}{ccc}
 H^0(Y - \bar{y}, \mathcal{O}_Y) & \xrightarrow{t_{\bar{y}}} & W = \prod_j \mathbb{C}((z_j^{1/e_1^{(j)}})) \times \cdots \times \mathbb{C}((z_j^{1/e_{k_j}^{(j)}})) \\
 \text{Tr}_X^Y \downarrow & & \downarrow \text{Tr} \\
 H^0(X - \bar{x}, \mathcal{O}_X) & \xrightarrow{t_x} & V = \prod_j \mathbb{C}((z_j))
 \end{array}$$

- $\text{Tr}(U) \subset U$ implies that $\text{Tr}(U) = U \cap \mathbb{C}((z))$ and, therefore, $\exists(X, x, t_x)$ such that $\text{Tr}(U) = t_x(H^0(X - x, \mathcal{O}_X))$.
- The inclusion $\text{Tr}(U) \subset U$ induces the morphism $\pi : Y \rightarrow X$.

3.1. Finite dimensional orbits

Consider the set of geometric data

$$\{(Y, \sigma_Y, \bar{y}, t_{\bar{y}}, L, \phi_{\bar{y}})\} / \text{equiv}$$

where

$(Y, \bar{y}, t_{\bar{y}})$ as above,

σ_Y an order p automorphism, $\sigma_Y(y_i) = y_{i+1}$, $\sigma_Y(y_p) = y_1$,

$(L, \phi_{\bar{y}})$ line bundle with a trivialization along $\bar{y}$

The Krichever map is

$$(Y, \sigma_Y, \bar{y}, t_{\bar{y}}, L, \phi_{\bar{y}}) \mapsto (t_{\bar{y}} \circ \phi_{\bar{y}}) H^0(Y - \bar{y}, L) \subset V$$

Γ_V acts on $\text{Gr}(V)^p := \text{Gr}(V) \times \dots \times \text{Gr}(V)$ by homotheties (componentwise)

$$\mu^p: \Gamma_V \times \text{Gr}(V)^p \longrightarrow \text{Gr}(V)^p$$

$$(g, (U_1, \dots, U_p)) \mapsto (gU_1, \dots, gU_p)$$

3.2. Finite dimensional orbits

For $U \in \text{Gr}(V)$ we define U_σ to be $(U, \sigma(U), \dots, \sigma^{p-1}(U))$ and let $\Gamma_V(U_\sigma)$ be its orbit under the action of Γ_V .

Theorem. *The following conditions are equivalent.*

1. $\dim_{\mathbb{C}} T_{U_\sigma}(\Gamma_V(U_\sigma)/\bar{\Gamma}_V^+) < \infty$,
2. *there exist data $(C, \sigma_C, \bar{y}, t_y, L, \phi_y)$ such that its image by the Krichever map is U .*

Proof. *Follows from*

- *If $d\mu_{U_\sigma}^p$ is the differential of $\Gamma_V \rightarrow \text{Gr}(V)^p$, then*

$$T_{U_\sigma}(\Gamma_V(U_\sigma)/\bar{\Gamma}_V^+) \simeq T_1\Gamma_V/(\ker d\mu_{U_\sigma}^p + T_1\bar{\Gamma}_V^+),$$

- *Let $A_U := \{g \in V \mid g \cdot U \subseteq U\}$ be the “stabilizer”, then*

4.1. Geometric interpretation

Fix data $(Y, \bar{y}, t_{\bar{y}})$ where

Y a projective irred. smooth irreducible curve

$\bar{y} = \{y_1, \dots, y_p\}$ pairwise distinct smooth points

$t_{\bar{y}} = \{t_{y_1}, \dots, t_{y_p}\}$ isomorphisms $t_{y_j} : \hat{\mathcal{O}}_{Y, y_j} \xrightarrow{\sim} \mathbb{C}[[z_j]]$

Given $L \in J_{g-1}(Y)$, there are canonical commutative diagrams:

$$\begin{array}{ccc}
 \hat{C}_V \hookrightarrow & \xrightarrow{\alpha} & Y \\
 \phi_V \downarrow & & \downarrow \phi_L \\
 \Gamma_V^- & \xrightarrow{\gamma} & J_{g-1}(Y)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \Gamma_V & \xrightarrow{\hat{\gamma}} & J_{g-1}^\infty(Y, \bar{y}) \hookrightarrow \text{Gr}(V) \\
 \downarrow & & \downarrow \pi \\
 \Gamma_V / \bar{\Gamma}_V^+ & \xrightarrow{\quad} & J_{g-1}(Y)
 \end{array}$$

with $J_{g-1}^\infty(Y, \bar{y}) := \{(L, \phi) \text{ s.t. } L \in J_{g-1}(Y) \text{ and } \phi : \hat{L}_{\bar{y}} \xrightarrow{\sim} \hat{\mathcal{O}}_{Y, \bar{y}}\}$.

It holds

$$\boxed{\text{Kr}^* \text{Det}_V^* \simeq \pi^* \mathcal{O}_{J_{g-1}}(\Theta)}$$

and therefore, one expects a relation between the τ -function and the theta function of $J_{g-1}(Y)$.

4.2. Geometric interpretation

$\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\}$ symplectic basis for $H_1(Y, \mathbb{Z})$.

$\{\omega_1, \dots, \omega_g\}$ holomorphic 1-forms with $\int_{\alpha_i} \omega_j = \delta_{ij}$.

Ω the period matrix and $\theta(z)$ the theta function.

$A^{(j)} : H^0(Y, \omega_Y) \rightarrow \mathbb{C}[[z_j]]dz_j$ maps ω_i to its germ at x_j , and let

$$A(t) := A^{(1)}t^{(1)} + \dots + A^{(p)}t^{(p)}$$

where $t = (t^{(1)}, \dots, t^{(p)})$ and $t^{(j)} = (t_1^{(j)}, t_2^{(j)}, \dots)$.

For $j \in \{1, \dots, p\}$ and $n > 0$, let $\eta_n^{(j)}$ be defined by

$$\int^x \eta_n^{(j)} = z_j^{-n} - 2 \sum_{m=1}^{\infty} q_{nm}^{(j)} \frac{z_j^m}{m} \quad \text{at } x = x_j .$$

and consider

$$Q(t) := \sum q_{nm}^{(1)} t_n^{(1)} t_m^{(1)} + \dots + \sum q_{nm}^{(p)} t_n^{(p)} t_m^{(p)}$$

4.3. Geometric interpretation

Define the τ -function as

$$\tau(\xi, t) := \exp(Q(t)) \theta(A(t) + \xi)$$

$(\xi \in \mathbb{C}^g)$.

Fact:

$$\tau_U(t) = \tau(\xi, t)$$

Theorem. *BA functions (introduced from the above geometric data) satisfy the $(1, \dots, 1)$ -KP-hierarchy*

$$\text{Res}_{z=0} \text{Tr} \left(\frac{\psi_{u,\xi}(z, t)}{(1, \dots, z_u, \dots, 1)} \cdot \frac{\psi_{v,\xi}^*(z, s)}{(1, \dots, z_v, \dots, 1)} \right) dz = 0$$

5.1. Schottky Problem

Let $X := \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)$ an irreducible p.p.a.v. . Let ξ be in $\mathbb{C}^g$. Let θ_X be its theta function.

Let $A^{(1)}, \dots, A^{(p)}$ be $g \times \infty$ -matrices of rank g and let $Q^{(1)}, \dots, Q^{(p)}$ be symmetric quadratic forms.

One introduces the τ -function by

$$\tau(\xi, t) := \exp(Q(t)) \theta(A(t) + \xi)$$

and BA-functions, $\psi_{u,\xi}(z, t)$.

Theorem. *The following conditions are equivalent.*

1. *There exists $(C, \bar{x}, t_{\bar{x}})$ s.t. $X_{\Omega} \simeq J(C)$ as p.p.a.v. .*
2. *There exist $A^{(1)}, \dots, A^{(p)}$ and $Q^{(1)}, \dots, Q^{(p)}$ s.t. $\tau(\xi, t)$ is a τ -function of the $(1, \dots, 1)$ -KP-hierarchy for all $\xi \in \mathbb{C}^g$.*

Proof. 1) $\Rightarrow$ 2) *It follows from the section 2.*

5.2. Schottky Problem

2) $\Rightarrow$ 1)

1. Since $\tau(\xi, t)$ satisfies the $(1, .^p., 1)$ –KP-hierarchy,

$$\exists U_\xi \in \text{Gr}(V) \text{ s.t. } \tau(\xi, t) = \tau_{U_\xi}(t)$$

$$\varphi: \mathbb{C}^g \longrightarrow \text{Gr}(V)$$

$$\xi \longmapsto U_\xi .$$

This map factorizes and gives $X \hookrightarrow \text{Gr}(V)$.

2. A is a surjective map $T_1 \Gamma_V^- \rightarrow T_\xi X$ and yields

$A_\xi: \Gamma_V^- \longrightarrow \hat{X}_\xi$. The properties of τ imply

$$\begin{array}{ccc} \Gamma_V^- & \xrightarrow{\mu_\xi} & \Gamma(U_\xi)/\bar{\Gamma}_V^+ \\ & \searrow A_\xi & \nearrow \\ & \hat{X}_\xi & \end{array}$$

hence $\dim T_{U_\xi} \Gamma(U_\xi)/\bar{\Gamma}_V^+$ is finite. Then, $\exists(C_\xi, \bar{x}_\xi, t_\xi, L_\xi, \phi_\xi)$ associated to U_ξ . Further, $(C_\xi, \bar{x}_\xi, t_\xi)$ does not depend on ξ .

5.3. Schottky Problem

3. Since $\text{rk } A^{(j)} = g$, that the image of

$$X \hookrightarrow \overline{\text{Pic}}(C)$$

$$\xi' \longmapsto L_{\xi'}$$

is $\text{Pic}^0(C) \cdot L_{\xi}$. So $\text{Pic}^0(C)$ is a quotient of an abelian variety $\implies C$ is smooth complete and of genus $\leq g$.

4. By construction, the τ -functions of X and of $J(C)$ coincide. Hence

$$\theta_X(A(t) + \xi) = \exp(q(t)) \theta_J(A_J(t) + \xi)$$

($q(t)$ is a quadratic function). Therefore $\Theta_X = \Theta_J$ and $X \xrightarrow{\sim} \text{Pic}^0(C)$ as p.p.a.v.'s. In particular, the curve C is irred. and of genus g .

5.4. Schottky Problem

Main Theorem. *Let X be an irreducible p.p.a.v. of dimension $g > 1$. Then the following conditions are equivalent.*

1. *$\exists C$, genus g projective irreducible smooth curve, with an order p automorphism $\sigma_C : C \rightarrow C$ such that $X \simeq J(C)$ as a p.p.a.v.*
2. *There exist $g \times \infty$ -matrices $A^{(1)}, \dots, A^{(p)}$ of rank g and symmetric quadratic forms $Q^{(1)}, \dots, Q^{(p)}$, such that*
 - a) *$\exists \xi_0 \in \mathbb{C}^g$, the corresponding BA-functions satisfy the $(1, \dots, 1)$ -KP hierarchy*

$$\text{Res} \left(\sum_{j=1}^p z^{-\delta_{ju} - \delta_{jv}} \psi_{u, \xi_0}^{(j)}(z, t) \psi_{v, \xi_0}^{*(j)}(z, s) \right) dz = 0$$

- b) *$\exists \xi_1 \in \mathbb{C}^g$ (depending on ξ_0) such that*

$$\text{Res} \left(\sum_{j=1}^p z^{-\delta_{ju} - \delta_{jv}} \psi_{v+1, \xi_0}^{(j+1)}(z, \sigma^*(t)) \psi_{u, \xi_1}^{*(j)}(z, s) \right) dz = 0$$

where $\sigma^*(t) := (t^{(p)}, t^{(1)}, t^{(2)}, \dots, t^{(p-1)})$.

5.5. Schottky Problem

Proof. 1) $\Rightarrow$ 2)

For any $\xi \in \mathbb{C}^g$ the corresponding BA-functions satisfies a).

The embedding $\text{Pic}(C) \hookrightarrow \text{Gr}(V)$ is equivariant

$$\implies \sigma(U_\xi) = U_{\sigma_C^* \xi}.$$

The relation between ψ_{u,ξ_0} and ψ_{v,ξ_1} for $\xi_1 = \sigma_C^* \xi_0$ implies b).

2) $\Rightarrow$ 1)

Similarly to Shiota, a) holds for ξ_0 iff holds for every ξ .

From a) and the previous Theorem, there exists a triple

$(C, \bar{x}, t_{\bar{x}})$ such that $X_\Omega \simeq J(C)$ as a p.p.a.v. .

Part b) says that $\sigma(U_{\xi_0})$ coincides with U_{ξ_1} , so the orbit

$\Gamma_V((U_{\xi_0})_\sigma) / \bar{\Gamma}_V^+$ is finite dimensional.

Section 3 implies that σ induces an automorphism on $\text{Pic}(C)$ and, therefore, of C .

6. Open Questions

- Determine what is the minimum number of equations that characterizes.
- Study moduli spaces of covers (Hurwitz spaces) in terms of infinite grassmannians and hierarchies of p.d.e. .
- Find similar characterizations for other cases of geometric relevance (e.g. Galois covers, action of a group G with prescribed ramification).