

Supersymmetric products of SUSY-curves †

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Abstract: The supersymmetric product of a SUSY-curve over a field is constructed with the aid of a theorem of invariants, and the notion of relative superdivisor is introduced. A universal superdivisor is defined in the supersymmetric product by means of Manin's superdiagonal, and it is proven that every superdivisor can be obtained in a unique way as a pullback of the universal superdivisor.

1. Introduction

1.1 Preliminaries on SUSY-curves

Super Riemann surfaces or *supersymmetric curves* (SUSY-curves) were introduced by Baranov, Schwarz [3] and Friedan [12] as an important tool for understanding the geometric structure of superstrings. The moduli space of these objects is necessary for the development of a Polyakov formalism for superstring theory.

As often happens in supermanifold theory, there are two different approaches to SUSY-curves; the one based on Rogers [30] and DeWitt's [11] formulation of supermanifolds, and the other based on Berezin-Leites-Kostant's [8], [22], [20] formulation of graded manifolds.

In the first approach, super Riemann surfaces, or superconformal manifolds as they are also often called, are — quoting directly from [31] — “(1,1)-dimensional complex supermanifolds composed of superdomains glued together by superconformal transformations (that is to say, by transformations preserving the operator $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$ up to a factor)”. In addition to the pioneering work by Baranov, Schwarz and Friedan, remarkable contributions to the geometry of super Riemann surfaces and their moduli space following this approach have been supplied, among others, by Crane, Rabin, Freund and Topiwala [10], [28], [29], and Baranov, Rosly, Schwarz and Voronov [3], [31], [32].

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In the second approach, closer to the spirit of Algebraic Geometry, graded Riemann surfaces are regarded as $(1, 1)$ -dimensional graded locally ringed spaces $(X, \mathcal{A})$ such that $(X, \mathcal{O} = \mathcal{A}/\mathcal{N})$ is a smooth proper algebraic curve ($\mathcal{N}$ denoting the nilpotent ideal of $\mathcal{A}$), and where the odd part $\mathcal{A}_1$ of $\mathcal{A}$ is a spin structure on $(X, \mathcal{O})$, i.e., a line bundle whose square is isomorphic with the canonical bundle. SUSY-families or (relative) SUSY-curves were defined as proper flat morphisms of relative dimension $(1, 1)$ endowed with a relative spin structure. In that context, a graded Riemann surface is a single SUSY-curve or a SUSY-family over a point (the spectrum of a field) [23]. There are many works in this direction, of very different depth and scope, considering either families or single SUSY-curves. Among others, those of Manin [23] [24], Batchelor and Bryant [6], Giddings and Nelson [13], [14], Bartocci, Bruzzo and Hernández Ruipérez [5], and LeBrun and Rothstein [21] — devoted to the construction of the moduli space of SUSY-curves as a superorbifold — might be cited.

This paper follows the second approach and should be considered as a first step towards an algebraic construction of a Jacobian supervariety of a single SUSY-curve $\mathcal{X}$ fulfilling the usual functorial requirements. Thus, in this paper by a SUSY-curve we shall understand a SUSY-family over a point, that is to say, a *graded Riemann surface*.

In order to establish these requirements more precisely, a survey of the most outstanding aspects of the corresponding theory for ordinary (non-graded) algebraic curve seems to be necessary. Although very interesting reviews of this topic can be found in [26] or [19], let us briefly recall in the following section some aspects of the diverse theories about the Jacobian variety of an algebraic curve, stressing those more relevant to SUSY-curves.

1.2. The Jacobian variety of an algebraic curve. A historical survey

One of the most prominent features of the theory of algebraic curves is the study of the Jacobian variety, i.e. the variety that parametrizes (modulo isomorphisms) the line bundles of degree g on the curve. There are many ways for constructing the Jacobian variety, differing either in method or in scope.

The first definition of the Jacobian variety $J(X)$ of a complex algebraic curve is of an analytic nature and is based on the period matrix. Once a 'canonical' basis $(\delta_1, \dots, \delta_{2g})$ of 1-cycles and a basis $(\omega_1, \dots, \omega_g)$ of holomorphic 1-forms on X have been chosen, $J(X)$ is defined as the complex torus $H^0(X, \Omega^1)/\Lambda$, where Λ is the lattice generated by the columns $\Pi_h = (\int_{\delta_h} \omega_1, \dots, \int_{\delta_h} \omega_g)$ of the period matrix. Then, if $\text{Div}^0(X)$ denotes the group of divisors of degree 0 of X , we have the so-called Abel map:

$$\begin{aligned} \mu: \text{Div}^0(X) &\rightarrow J(X) \\ \sum_{i=1}^g (q_i - p_i) &\mapsto \left(\int_{p_1}^{q_1} \omega_1, \dots, \int_{p_g}^{q_g} \omega_g \right) / \Lambda. \end{aligned}$$

Two classical theorems, namely Abel's theorem and the Jacobi inversion theorem, ensure that the Abel map induces an isomorphism

$$\bar{\mu}: \text{Div}^0(X)/\text{linear equivalence} \simeq J(X)$$

which is generically one-to-one, i.e., it is a birational map.

The second classical construction of the Jacobian variety is due to A. Weil [33], who defined $J(X)$ as a set by letting $J(X) = \text{Div}^0(X)/(\text{linear equivalence})$, thus shifting the problem to the determination of the structure of $J(X)$ as an abelian variety. This was done as follows: let $\text{Div}^g(X) = S^g X$ be the set of positive divisors of degree g of X , that is, unordered families of g points. If a point $x \in X$ is fixed, the Abel map

$$\begin{aligned} \mu^{(g)}: \text{Div}^g(X) &\rightarrow J(X) \\ x_1 + \dots + x_g &\mapsto [x_1 + \dots + x_g - gx], \end{aligned}$$

where g is the genus of X and $[]$ means the linear equivalence class, is shown to be one-to-one for generic x . Moreover, if a point $x \in X$ has been fixed, it is proved that, given two generic positive divisors D, D' of degree g , the divisor $D + D' - gx$ is linearly equivalent to a unique positive divisor $\bar{D}$ of degree g , thus allowing us to define a local group law in the Jacobian variety by letting

$$[D - gx] + [D' - gx] = [\bar{D} - gx].$$

The third procedure for the construction of the Jacobian variety is due to Chow [9], who noticed that for a great enough p the Abel map $\mu^{(p)}: \text{Div}^p(X) \rightarrow J(X)$ is a projective bundle, so that $J(X)$ can be endowed with a structure of projective algebraic group.

Inspired by this procedure, but with an enormous change in mentality, Grothendieck constructed the Picard schemes (the Jacobian of a curve is a particular case of a Picard scheme) as quotients of Hilbert schemes by the action of a flat equivalence relation [15], [1]. The philosophy of this method was the theory of the *functor of points*, which is based on the following principle: classically, the knowledge of the points of a variety allows us to determine the variety only as a set, and the algebraic or analytic structure has to be assigned separately. However, if we consider not only ordinary points, i.e., morphisms of a single point into the variety, but rather S -valued 'points', i.e. morphisms of a variety S into our variety, then the knowledge of all these 'points' characterizes the algebraic variety. The techniques for determining a scheme from the functor of its points are called 'construction techniques' and the theorems stating that a certain functor is actually the functor of points of a scheme are known as 'representability theorems'.

Let us explain briefly how these techniques apply to construction of the Picard scheme of an algebraic variety X . As a first step, we wish the points of the Picard scheme to be the line bundles on X . But, as we have seen, we need all the S -valued 'points' for every scheme S and not only the ordinary points. Since these 'points' should be, roughly speaking, 'families of line bundles' of X parametrized by S , Grothendieck defined the Picard functor of X by taking $\text{Pic}_X(S)$ as the set of line bundles on $X \times S$ modulo those obtained by pulling back line bundles on the parameter space S . In this way we get the candidate $\text{Pic}_X(S)$ for the set of S -valued 'points' of the Picard scheme. Under certain conditions, the Picard functor

is representable; that is, it is the functor of points of a well determined scheme, namely, the Picard scheme of X .

The method of the functor of points is not only as elegant as Grothendieck's ideas used to be, but is also extremely precise. It was exploited to show how even for smooth algebraic varieties, the Picard scheme may fail to be reduced. In fact the Picard scheme of the so-called Igusa surface is a scheme whose ring sheaf has nilpotent elements (see [15]).

Weil's construction of the Jacobian of a smooth algebraic curve was drawn to this framework by Iversen [17] and Artin [2]. Iversen showed that the functor of relative divisors of degree g of $X \times S \rightarrow S$, once properly defined, is the functor of points of the symmetric product $S^g X$ considered as the quotient of the cartesian product $X \times \dots \times X$ by the action of the symmetric group. Weil's techniques for defining an algebraic group structure on an algebraic variety which is birationally equivalent to an algebraic group (afterwards generalized to arbitrary schemes by Artin [2]) applies to this case to perform the Abel morphism, taking the quotient with respect to the linear equivalence relation in order to represent the Picard functor of the curve and thus constructing its Jacobian as an abelian variety.

1.3. Towards the Jacobian of a SUSY-curve. The supersymmetric product

Within the context of supermanifolds *à la* Rogers, the analytic construction of the Jacobian (the one based on the period matrix) has been generalized for super Riemann surfaces in [31] and [32]. This Jacobian is an ordinary (non-super) manifold, but the superanalogous of Abel's and Jacobi inversion theorems have been proved. In the same framework, a cohomological construction of the Picard variety for a particular kind of Roger supermanifolds, the DeWitt ones, has been made in [4].

However, as far as we know, there are no constructions of the Jacobian scheme of SUSY-curves within the algebraic approach, i.e. via graded ringed spaces. Our plan is to extend the Weil-Iversen construction to this case, and the first step to do so is to construct the supersymmetric product of a SUSY-curve and to determine its functor of points.

There are further reasons to adopt this procedure, based on the intrinsic interest of the supersymmetric product. In fact, regardless of the method used to construct the Jacobian, the structure of the symmetric products $S^p X$ and the diverse Abel morphisms $S^p X \rightarrow J(X)$ turn out to be a key point in the theory of Jacobian varieties (see, for instance, [18], [25]), and have proved to be a very important tool for the solution of the Schottky problem ([27]).

This paper is organized as follows. The supersymmetric product $S^g \mathcal{X}$ of a superscheme $\mathcal{X}$ of dimension $(1, 1)$ is constructed in Part 2 as the orbit ringed space obtained through the action of the symmetric group on the cartesian product of g copies of the supercurve, g being its genus. Actually, this supersymmetric product is a superscheme of dimension (g, g) ; this statement is far from trivial, and is equivalent to a theorem on invariants, namely, the fact that the invariant subalgebra of a certain exterior algebra is an exterior algebra as well. It should be

stressed that this result is no longer true for superschemes of higher odd dimension ($n > 1$).

Part 3 deals with the determination of the functor of points of the supersymmetric product. The main result here is that this functor is the functor of the so-called *relative superdivisors*. This means that there exists a universal relative superdivisor in $\mathcal{X} \times S^g \mathcal{X} \rightarrow S^g \mathcal{X}$ such that every relative superdivisor of $\mathcal{X} \times \mathcal{S} \rightarrow \mathcal{S}$ can be obtained in a unique way as the pullback of the universal one through a morphism $\mathcal{S} \rightarrow S^g \mathcal{X}$.

The construction of the universal superdivisor is inspired by Manin's definition of superdiagonal ([23], [7]) and by the fact — also pointed out by him — that for SUSY-curves relative superdivisors of $\mathcal{X} \times S^g \mathcal{X} \rightarrow S^g \mathcal{X}$ are not $\mathcal{S}$ -valued 'points', because $\mathcal{S}$ -points have codimension (1, 1) and not codimension (1, 0) as superdivisors do.

However, given an $\mathcal{S}$ -point, defined locally in relative conformal coordinates by the equations $z - z_0 = \theta - \theta_0 = 0$, one can associate to this point the relative superdivisor of degree 1 with the same support defined by the equation $z - z_0 - \theta\theta_0 = 0$, which is in fact independent of the coordinate system, and is therefore globally defined.

The $\mathcal{S}$ -point σ can be recovered from the function $f = z - z_0 - \theta\theta_0$, because the defining ideal $(z - z_0, \theta - \theta_0)$ for the point is exactly the ideal $(f, D(f))$, where $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$ is the local generator of the conformal structure.

This shows the nature of the equations defining relative divisors of degree 1. Relative divisors of degree g should be then defined 'generically' by products of equations of this kind. This can be stated in a precise way only for a suitable universal superdivisor and for those superdivisors obtained from it by pullback. This is the reason underlying our definition of relative superdivisor and our construction of the universal relative superdivisor.

This notion is related to some definitions given in [31]. In that reference, a principal zero of a meromorphic section s of a line bundle on a superconformal manifold is defined as a point P fulfilling $s(P) = 0$ and $Ds(P) = 0$, and the relationship between Cartier divisors, points, and principal zeroes of meromorphic sections in that context is discussed. Our definition will lead to an equivalent notion of a Cartier divisor if the base scheme $\mathcal{S}$ is taken as a (non-graded) single point.

As a final comment, let us say that the construction of the supersymmetric product of a SUSY-curve and the determination of its functor of points can be extended straightforwardly to SUSY-families parametrized by an *ordinary* scheme. The case of SUSY-families over an arbitrary superscheme will be considered in a forthcoming paper.

2. Supersymmetric products

2.1. Definitions

Let us start with the definition of *graded scheme* or *superscheme*. Let $\mathcal{X} = (X, \mathcal{A})$ be a graded ringed space, consisting of a topological space X and a sheaf $\mathcal{A}$ of $\mathbb{Z}_2$ -graded algebras.

Definition 1. A superscheme of dimension (m, n) over a field k is a graded ringed space $\mathcal{X} = (X, \mathcal{A})$ where $\mathcal{A}$ is a sheaf of graded k -algebras such that:

- 1) $(X, \mathcal{O})$ is an algebraic scheme of dimension m over k , where $\mathcal{O} = \mathcal{A}/\mathcal{J}$, $\mathcal{J}$ being the ideal $A_1 + A_1^2$.
- 2) $\mathcal{J}/\mathcal{J}^2$ is a locally free rank n $\mathcal{O}$ -module and $\mathcal{A}$ is locally isomorphic to $\bigwedge_{\mathcal{O}}(\mathcal{J}/\mathcal{J}^2)$.

$(X, \mathcal{O})$ will be called the underlying ordinary scheme.

Let us consider the product

$$\mathcal{X}^g = (X^g, \mathcal{A}^{\otimes g}) \quad (1)$$

where X^g denotes the cartesian product $X \times \dots \times X$, and $\mathcal{A}^{\otimes g} = \mathcal{A} \otimes \dots \otimes \mathcal{A}$.

The symmetric group S_g acts on $\mathcal{X}^g$ by graded automorphisms of superschemes according to the rule:

$$\begin{aligned} \sigma: X^g &\rightarrow X^g \\ (x_1, \dots, x_g) &\mapsto (x_{\sigma(1)}, \dots, x_{\sigma(g)}) \end{aligned} \quad (2)$$

on the space and:

$$\begin{aligned} \sigma^*: \mathcal{A}^{\otimes g} &\rightarrow \sigma_* \mathcal{A}^{\otimes g} \\ f_1 \otimes \dots \otimes f_g &\mapsto \prod_{\substack{i < j \\ \sigma(i) > \sigma(j)}} (-1)^{|f_i||f_j|} f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(g)} \end{aligned} \quad (3)$$

on the sheaves. This action reduces to the ordinary action of S_g on the scheme $(X^g, \mathcal{O}^{\otimes g})$. Then, we have the orbit space $S^g X$, a natural projection $p: X^g \rightarrow S^g X$, and an invariant sheaf $\mathcal{O}_g = (\mathcal{O}^{\otimes g})^{S_g}$ on $S^g X$, whose sections on an open subset $V \subset S^g X$ are

$$\mathcal{O}_g(V) = \{f \in \mathcal{O}^{\otimes g}(p^{-1}(V)) \mid \sigma^* f = f \text{ for every } \sigma \in S_g\}. \quad (4)$$

It is well-known that if $(X, \mathcal{O})$ is a projective scheme, the ringed space $(S^g X, \mathcal{O}_g)$ is a scheme, the g -th symmetric product of $(X, \mathcal{O})$.

Let us consider the sheaf $\mathcal{A}_g = (\mathcal{A}^{\otimes g})^{S_g}$ of graded invariants on $S^g X$ defined as above by letting:

$$\mathcal{A}_g(V) = \{f \in \mathcal{A}^{\otimes g}(p^{-1}(V)) \mid \sigma^* f = f \text{ for every } \sigma \in S_g\} \quad (5)$$

for every open subset $V \subset S^g X$.

2.2. The case of supercurves

Definition 2. A proper smooth supercurve is a superscheme of dimension $(1, n)$, such that the underlying ordinary scheme is a proper smooth algebraic curve over a field k .

Let $\mathcal{X}$ be a smooth proper supercurve. As $(X, \mathcal{O})$ is projective (because it has very ample sheaves), the symmetric product $(S^g X, \mathcal{O}_g)$ is a scheme (in fact, it is smooth, which is no longer true for a higher dimensional X).

Theorem 1. Let $\mathcal{X} = (X, \mathcal{A})$ be a smooth proper supercurve. The graded ringed space $S^g \mathcal{X} = (S^g X, \mathcal{A}_g)$ is a superscheme if and only if $n = 1$, that is, if and only if $\mathcal{X}$ is a superscheme of dimension $(1, 1)$.

In that case, $S^g \mathcal{X}$ will be called the g -th supersymmetric product of $\mathcal{X}$.

2.2.1. Sketch of the proof

There is a natural projection $\mathcal{A}_g \rightarrow \mathcal{O}_g$, and we have only to ascertain whether $\mathcal{A}_g$ is locally the exterior algebra of a certain locally free $\mathcal{O}_g$ -module. We can assume that $\mathcal{A} = \bigwedge_{\mathcal{O}}(\mathcal{N})$, $\mathcal{N}$ being a free rank n $\mathcal{O}$ -module.

Let us write $\mathcal{N}_i = \mathcal{O} \otimes \dots \otimes \overset{1i)}{\mathcal{N}} \otimes \dots \otimes \mathcal{O}$ and $\mathcal{M} = \mathcal{N}_1 \oplus \dots \oplus \mathcal{N}_g$. Now, if we write $\bar{\mathcal{O}} = \mathcal{O}^{\otimes g}$ and $\bar{\mathcal{A}} = \mathcal{A}^{\otimes g}$, we have:

$$\bar{\mathcal{A}} = \bigwedge_{\mathcal{O}}(\mathcal{N}) \otimes_{\mathcal{O}} \dots \otimes_{\mathcal{O}} \bigwedge_{\mathcal{O}}(\mathcal{N}) \simeq \bigwedge_{\bar{\mathcal{O}}}(\mathcal{M}). \quad (6)$$

The symmetric group S_g acts on $\mathcal{M}$ by $\sigma(n_1 + \dots + n_g) = n_{\sigma(1)} + \dots + n_{\sigma(g)}$ and this action provides an action $\sigma: \bar{\mathcal{A}} \rightarrow \bar{\mathcal{A}}$ on the exterior algebra $\bar{\mathcal{A}} = \bigwedge_{\bar{\mathcal{O}}}(\mathcal{M})$, given by $\sigma(m_1 \wedge \dots \wedge m_p) = \sigma(m_1) \wedge \dots \wedge \sigma(m_p)$.

Thus, we have two actions of S_g on $\bar{\mathcal{A}}$, the one just defined, and that previously introduced in (3). However, it turns out that the two actions are the same, because they coincide on $\bigwedge_{\bar{\mathcal{O}}}^1(\mathcal{M}) = \mathcal{M}$ and are morphisms of graded algebras.

Let us denote by $\mathcal{M}^{S_g}$ the $\mathcal{O}_g$ -module consisting of the invariant sections of $\mathcal{M}$. Our main theorem now follows from:

Lemma 1. The natural morphism of sheaves of graded $\mathcal{O}_g$ -algebras over $S^g X$,

$$\phi: \bigwedge_{\mathcal{O}_g}(\mathcal{M}^{S_g}) \rightarrow (\bigwedge_{\mathcal{O}}(\mathcal{M}))^{S_g} = \mathcal{A}_g, \quad (7)$$

is an isomorphism if and only if $n = 1$.

Proof. The proof is a calculus of invariants in the exterior algebra of a free module over a commutative ring, which allows us to use standard methods of Commutative Algebra. We shall proceed by steps.

Let us first notice that the statement is invariant by completion at the maximal points of $\mathcal{O}_g$. This reduces the problem to the case of the polynomial ring $\mathcal{O} = k[x]$.

Now, for every $0 \leq p \leq g$ we have:

$$\bigwedge^p(\mathcal{M}) = \bigoplus_{i_1 < \dots < i_p} \mathcal{N}_{i_1} \wedge \dots \wedge \mathcal{N}_{i_p} \quad (8)$$

and S_g acts transitively by permutation of summands.

In fact, $\mathcal{N}_{i_1} \wedge \dots \wedge \mathcal{N}_{i_p} = \sigma_{i_1 \dots i_p}(\mathcal{N}_1 \wedge \dots \wedge \mathcal{N}_p)$, $\sigma_{i_1 \dots i_p} \in S_g$ being any permutation of type $\sigma_{i_1 \dots i_p} = \begin{pmatrix} 1 & \dots & p & \dots \\ i_1 & \dots & i_p & \dots \end{pmatrix}$.

Then, an invariant element $m = \sum_{i_1 < \dots < i_p} n_{i_1} \wedge \dots \wedge n_{i_p}$ is characterized by $n_1 \wedge \dots \wedge n_p$, and we have an isomorphism:

$$\begin{aligned} (\mathcal{N}_1 \wedge \dots \wedge \mathcal{N}_p)^{S_p \times S_{g-p}} &\simeq (\bigwedge^p(\mathcal{M}))^{S_g} \\ n_1 \wedge \dots \wedge n_p &\mapsto \sum_{\sigma \in S_g / (S_p \times S_{g-p})} \sigma(n_1 \wedge \dots \wedge n_p) \end{aligned} \quad (9)$$

where $S_p \times S_{g-p}$ denotes the subgroup of S_g consisting of the permutations leaving the subset $\{1, \dots, p\}$ invariant. In particular, $(\mathcal{N}_1)^{S_{g-1}} \simeq \mathcal{M}^{S_g}$.

Let us now assume that $n = 1$. Then

$$\mathcal{N}_1 \wedge \dots \wedge \mathcal{N}_p = \bar{\mathcal{O}} \cdot e_1 \wedge \dots \wedge e_p, \quad (10)$$

and we have:

$$(\mathcal{N}_1 \wedge \dots \wedge \mathcal{N}_p)^{S_p \times S_{g-p}} \simeq \bar{\mathcal{O}}^{-S_p \times S_{g-p}} \quad (11)$$

where $\bar{\mathcal{O}}^{-S_p \times S_{g-p}}$ stands for the subset of those $f \in \bar{\mathcal{O}}$ such that $(\sigma \times \mu)(f) = \text{sign}(\sigma) \cdot f$ for every $(\sigma \times \mu) \in S_p \times S_{g-p}$. In particular,

$$\mathcal{N}_1^{S_{g-1}} \simeq \bar{\mathcal{O}}^{S_{g-1}}. \quad (12)$$

The original morphism

$$\phi_p: \bigwedge_{\mathcal{O}_g}^p(\mathcal{M}^{S_g}) \rightarrow (\bigwedge_{\bar{\mathcal{O}}}^p(\mathcal{M}))^{S_g} = \mathcal{A}_g \quad (13)$$

is now a morphism

$$\bar{\phi}_p: \bigwedge^p \bar{\mathcal{O}}^{S_{g-1}} \rightarrow \bar{\mathcal{O}}^{-S_p \times S_{g-p}} \quad (14)$$

described by

$$\bar{\phi}_p(f_1 \wedge \dots \wedge f_p) = \sum_{\mu \in S_p} \text{sign}(\mu) \sigma_{\mu(1)}(f_1) \dots \sigma_{\mu(p)}(f_p) \quad (15)$$

where σ_i is the transposition of 1 and i .

A lengthy calculation involving symmetric functions allows us to prove that $\bar{\phi}_p$ is an isomorphism. We can also prove, basically because there exist invariant elements in the tensor product that cannot be written as tensor products of invariant elements, that ϕ_p is not an isomorphism when $n > 1$, thus finishing the sketch of the proof. $\square$

Moreover, if $\mathcal{X}$ is a proper smooth supercurve of odd dimension 1, the ring sheaf of the symmetric product $S^g \mathcal{X}$ is globally an exterior algebra. This is due to the

following fact. The natural epimorphism $\mathcal{A} \rightarrow \mathcal{O}$ induces an isomorphism $\mathcal{A}_0 \simeq \mathcal{O}$ between the even part of $\mathcal{A}$ and $\mathcal{O}$. The odd part is an invertible $\mathcal{O}$ -module $\mathcal{L}$, so that $\mathcal{A} = \mathcal{O} \oplus \mathcal{L} \simeq \bigwedge_{\mathcal{O}}(\mathcal{L})$. Then there is a canonical isomorphism:

$$\bigwedge_{\mathcal{O}_g}(\mathcal{M}^{S_g}) \simeq \mathcal{A}_g \quad (16)$$

where $\mathcal{M} = \bigoplus_{i=1}^g (\mathcal{O} \otimes \dots \otimes \overset{1i)}{\mathcal{L}} \otimes \dots \otimes \mathcal{O})$.

3. The functor of points of the Supersymmetric product

3.1. SUSY-curves

Henceforth calligraphic types will be reserved to graded ringed spaces and the structure ring sheaf of any ringed space will be denoted by $\mathcal{O}$ with the name of the ringed space as a subscript. Thus, for instance, $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ or simply $\mathcal{X}$ will mean a graded ringed space, whereas $(X, \mathcal{O}_X)$ or X will stand for the underlying ordinary ringed space.

Definition 3. A *supersymmetric curve* or SUSY-curve over a field k is a smooth proper supercurve $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ of odd dimension 1, endowed with a locally free submodule $\mathcal{D}$ of rank $(0, 1)$ of the tangent sheaf $T_{\mathcal{X}} = \text{Der}(\mathcal{O}_{\mathcal{X}})$ such that the composition of maps

$$\mathcal{D} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D} \xrightarrow{[,] } \text{Der}(\mathcal{O}_{\mathcal{X}}) \rightarrow \text{Der}(\mathcal{O}_{\mathcal{X}})/\mathcal{D} \quad (17)$$

is an isomorphism of $\mathcal{O}_{\mathcal{X}}$ -modules (see for instance [21]). $\mathcal{D}$ is called the *conformal structure* of the SUSY-curve

If $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}}, \mathcal{D})$ is a SUSY-curve, X can be covered by affine open subsets $U \subset X$ with local relative coordinates (z, θ) such that $\mathcal{D}$ is locally generated by $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$. These coordinates are called *conformal*.

There is a natural isomorphism $\mathcal{D}^* \simeq \text{Ber}(\mathcal{O}_{\mathcal{X}})$ and a 'Berezinian differential':

$$\partial: \Omega_{\mathcal{X}}^1 \rightarrow \text{Ber}(\mathcal{O}_{\mathcal{X}}) \simeq \mathcal{D}^* \quad (18)$$

which is nothing but the natural projection induced by the immersion $\mathcal{D} \rightarrow T_{\mathcal{X}}$. In conformal coordinates δ is described by $\partial(df) = \left[dz \otimes \frac{\partial}{\partial \theta} \right] \cdot D(f)$ where $\left[dz \otimes \frac{\partial}{\partial \theta} \right]$ denotes the local basis of $\text{Ber} \mathcal{O}_{\mathcal{X}}$ determined by (z, θ) (see [16]).

3.2. The universal divisor for an algebraic curve

This section is devoted to summarize the theory of the variety of divisors and the universal divisor for an (ordinary) algebraic curve X , also explaining how the relevant constructions and theorems can be recast if the variety of parameters is a superscheme. Suitable references are [15] or [17].

In this case, divisors of degree g are unordered families of g points, so that they are parametrized by the symmetric product $S^g X$. This can be made precise by means of the notion of relative divisor. If S is another scheme, relative divisors of $X \times S \rightarrow S$ of degree g , or families of divisors of X depending on a scheme of parameters S , are simply subschemes $Z \rightarrow X$ such that $\mathcal{O}_Z$ is a locally free $\mathcal{O}_S$ -module of rank g . There is a nice relative divisor Z_U of degree g of $X \times S^g X \rightarrow S^g X$, whose fibre at a point $(x_1, \dots, x_g) \in S^g X$ is the divisor $x_1 + \dots + x_g$ of X defined by it. Z_U is called the *universal divisor* because the map

$$\begin{aligned} \text{Hom}(S, S^g X) &\rightarrow \text{Div}_S^g(X \times S) \\ \phi &\mapsto (1 \times \phi)^{-1}(Z_U), \end{aligned} \quad (19)$$

where $\text{Div}_S^g(X \times S)$ denotes the set of relative divisors of degree g , is one-to-one. This means that each divisor can be obtained as a pullback of the universal divisor, which statement is known as the *representability theorem* for the symmetric product.

This theory is still viable when a superscheme is allowed as the space of parameters. The corresponding notion of relative divisor is:

Definition 4. Let X be an ordinary smooth curve and $(S, \mathcal{O}_S)$ a superscheme. A relative divisor of degree g of $X \times S \rightarrow S$ is a closed sub-superscheme Z of $X \times S$ of codimension $(1, 0)$ defined by a homogeneous ideal J of $\mathcal{O}_{X \times S}$ such that $\mathcal{O}_{X \times S}/J$ is a locally free $\mathcal{O}_S$ -module of rank $(g, 0)$.

The representability theorem now reads:

Theorem 2. Let X be a smooth proper curve over a field k and Z_U the universal divisor. The map

$$\begin{aligned} \text{Hom}(S, S^g X) &\rightarrow \text{Div}_S^g(X \times S) \\ \phi &\mapsto (1 \times \phi)^{-1}(Z_U), \end{aligned} \quad (20)$$

where $\text{Div}_S^g(X \times S)$ denotes the set of relative divisors of degree g , is one-to-one for every superscheme S .

The argument proving the representability theorem for ordinary schemes applies to this case only with minor changes. Two key points are then involved in the proof; the first is the construction of the universal divisor. If $\pi_i: X^g \rightarrow X$ is the i th projection and Δ_i is the relative divisor of $X \times X^g \rightarrow X^g$ obtained by pulling back the diagonal $\Delta \subset X \times X$ through $1 \times \pi_i: X \times X^g \rightarrow X \times X$, one can prove that there exists a unique relative divisor Z_U of $X \times S^g X \rightarrow S^g X$ such that

$$(1 \times p)^* Z_U = \Delta_1 + \dots + \Delta_g, \quad (21)$$

where $p: X^g \rightarrow S^g X$ is the natural projection. This divisor Z_U is the universal divisor and is locally described by the equation

$$(z - z_1) \cdots (z - z_g) = z^g - s_1 z^{g-1} + \dots + (-1)^g s_g = 0, \quad (22)$$

where z is a local parameter on X , $z_i = 1 \otimes \dots \otimes \overset{\downarrow i)}{z} \otimes \dots \otimes 1$, and $s_1, \dots, s_g$ are the symmetric functions of $z_1, \dots, z_g$.

The second key point is the 'determinant morphism' $\mathcal{S} \rightarrow S^g Z$, Z being a relative divisor of degree g , which deserves this name because its composition with $S^g Z \rightarrow S^g X$ provides the inverse mapping of (20) (See [17]). The determinant morphism for the locally free $\mathcal{O}_S$ -module $\mathcal{O}_Z$ of rank $(g, 0)$ is defined as follows. Each element b in the invariant sheaf $(\mathcal{O}_Z)_g = (\mathcal{O}_Z^{\otimes g})^{S_g}$ acts on the $\mathcal{O}_S$ -module $\bigwedge_{\mathcal{O}_S} \mathcal{O}_Z$ of rank $(1, 0)$ as the multiplication by a well-determined element $\det(b)$ in $\mathcal{O}_S$. This gives rise to a morphism of sheaves $(\mathcal{O}_Z)_g \rightarrow \mathcal{O}_S$, and thus to a morphism of schemes $\mathcal{S} \rightarrow S^g Z$.

The determinant morphism provides the inverse mapping of (20) because if b is an even element in $\mathcal{O}_Z$, $b_i = 1 \otimes \dots \otimes \overset{\downarrow i)}{b} \otimes \dots \otimes 1 \in \mathcal{O}_Z^{\otimes g}$, and we denote by $s_i(b)$ the symmetric functions of $b_1, \dots, b_g$, we have that

$$a_i = \det(s_i(b)) \quad (i = 1, \dots, g), \quad (23)$$

where $z^g - a_1 z^{g-1} + \dots + (-1)^g a_g$ is the characteristic polynomial of b acting on $\mathcal{O}_Z$ by multiplication.

3.3. Superdivisors on a SUSY-curve and universal superdivisor

Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}, \mathcal{D})$ be a SUSY-curve, $(X, \mathcal{O}_X)$ the ordinary underlying curve and $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ a superscheme.

Every closed sub-superscheme $\mathcal{Z}$ of $\mathcal{X} \times \mathcal{S} \rightarrow \mathcal{S}$ induces a closed sub-superscheme $\hat{\mathcal{Z}} = \mathcal{Z} \times_{\mathcal{X}} X$ of $X \times \mathcal{S} \rightarrow \mathcal{S}$ by reduction through the canonical immersion $X \rightarrow \mathcal{X}$.

Definition 5. A relative superdivisor of degree g of $\mathcal{X} \times \mathcal{S} \rightarrow \mathcal{S}$ is a closed sub-superscheme $\mathcal{Z}$ of $\mathcal{X} \times \mathcal{S}$ of codimension $(1, 0)$ whose reduction $\hat{\mathcal{Z}} = \mathcal{Z} \times_{\mathcal{X}} X$ is a relative divisor of degree g of $X \times \mathcal{S} \rightarrow \mathcal{S}$ (definition 4).

Let us describe relative superdivisors in local terms.

Lemma 3. A closed sub-superscheme $\mathcal{Z}$ of $\mathcal{X} \times \mathcal{S}$ of codimension $(1, 0)$ defined by an homogeneous ideal J of $\mathcal{O}_{\mathcal{X} \times \mathcal{S}}$ is a relative superdivisor of degree g if and only if $\mathcal{O}_{\mathcal{Z}} = \mathcal{O}_{\mathcal{X} \times \mathcal{S}}/J$ is a locally free $\mathcal{O}_{\mathcal{S}}$ -module of dimension (g, g) and, once a system of conformal coordinates (z, θ) has been chosen, J can be locally generated by an element of the form

$$f = z^g - (a_1 + \theta b_1)z^{g-1} + \dots + (-1)^g(a_g + \theta b_g), \quad (24)$$

where the a_i 's are even and the b_j 's odd elements in $\mathcal{O}_S$.

Now, in order to parametrize superdivisors by means of the supersymmetric product $S^g \mathcal{X} = (S^g X, \mathcal{O}_{S^g \mathcal{X}})$, we need a suitable definition of universal superdivisor, which is obtained in terms of Manin's superdiagonal. If Δ denotes the ideal of the diagonal immersion $\delta: \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$, we have that the kernel of the composition $\delta_* \Omega^1 \rightarrow \delta_* \Omega^1_{\mathcal{X}} \simeq \Delta/\Delta^2 \xrightarrow{\partial} \text{Ber}(\mathcal{O}_{\mathcal{X}})$ (see (18)) is a homogeneous ideal I of $\mathcal{O}_{\mathcal{X} \times \mathcal{X}}$, thus defining a sub-superscheme Δ^s called the *superdiagonal*.

Lemma 4. (Manin, [23]) *The superdiagonal $\Delta^s = (X, \mathcal{O}_{\mathcal{X} \times \mathcal{X}}/I)$ is a closed sub-superscheme of codimension $(1, 0)$ (a superdivisor). In conformal coordinates (z, θ) , it can be described by the equation:*

$$z_1 - z_2 - \theta_1 \theta_2 = 0 \quad (25)$$

where, as usual, $z_1 = z \otimes 1$ and $z_2 = 1 \otimes z$.

The universal superdivisor for SUSY-curves is now constructed as in the non-graded case. Let $p_i: \mathcal{X}^g \rightarrow \mathcal{X}$ be the i -th projection and let Δ_i^s be the sub-superscheme of $\mathcal{X} \times \mathcal{X}^g$ obtained by pulling the superdiagonal Δ^s back through $(1 \times p_i): \mathcal{X} \times \mathcal{X}^g \rightarrow \mathcal{X} \times \mathcal{X}$. Let us consider the sub-superscheme $\Delta_1^s + \dots + \Delta_g^s$ defined by the intersection ideal.

Theorem 3. *There is a unique closed sub-superscheme of codimension $(1, 0)$, $\mathcal{Z}^s$ of $\mathcal{X} \times S^g \mathcal{X} \rightarrow S^g \mathcal{X}$ such that*

$$(1 \times p)^* \mathcal{Z}^s = \Delta_1^s + \dots + \Delta_g^s, \quad (26)$$

where $p: \mathcal{X}^g \rightarrow S^g \mathcal{X}$ is the natural projection. Moreover, $\mathcal{Z}^s$ is a relative superdivisor of degree g , that will be called the *Universal Superdivisor*.

Proof. The first part of the theorem holds because in local conformal coordinates the equation of $\mathcal{Z}^s$ is

$$(z - z_1 - \theta \theta_1) \cdot \dots \cdot (z - z_g - \theta \theta_g) = 0, \quad (27)$$

which is a polynomial in (z, θ) with coefficients in the invariant sheaf $(\mathcal{O}_{\mathcal{X}})_g$, since:

$$\begin{aligned} (z - z_1 - \theta \theta_1) \cdot \dots \cdot (z - z_g - \theta \theta_g) &= \prod_{i=1}^g (z - z_i) - \theta \sum_{i=1}^g \theta_i \prod_{j \neq i} (z - z_j) \\ &= z^g + \sum_{h=1}^g (-1)^h (s_h + \theta \varsigma_h) z^{g-h}; \end{aligned} \quad (28)$$

here the s_h 's are the symmetric functions of $(z_1, \dots, z_g)$, and $\varsigma_h = \sum_{i=1}^g \sigma_i(\theta_1 \bar{s}_{h-1})$ are the odd symmetric functions, $\bar{s}_h$ being the symmetric functions of $(z_2, \dots, z_g)$ and σ_i any permutation of $(1, \dots, g)$ sending 1 to i , as in (9).

The second part follows directly from lemma 3. $\square$

Remark. The coefficients of the universal divisors enjoy an important property: if $\bar{D} = \sum_{i=1}^g (\theta_i \frac{\partial}{\partial z_i} + \frac{\partial}{\partial \theta_i})$ is the local derivation of $S^g \mathcal{X}$ induced by $D = \theta \frac{\partial}{\partial z} + \frac{\partial}{\partial \theta}$, then

$$\varsigma_i = \bar{D}(s_i). \quad (29)$$

3.4 The Representability Theorem for superdivisors

Let $(\mathcal{X}, \mathcal{D})$ be a SUSYcurve. For every superscheme $\mathcal{S}$ let us denote by

$$\mathcal{S} \rightarrow \text{Div}_S^g(\mathcal{X} \times \mathcal{S}) \quad (30)$$

the functor of relative superdivisors of degree g of $\mathcal{X} \times \mathcal{S} \rightarrow \mathcal{S}$.

Let us also consider the functor of points of the supersymmetric product, defined by:

$$\mathcal{S} \rightarrow \text{Hom}(\mathcal{S}, S^g \mathcal{X}). \quad (31)$$

Since $\mathcal{O}_{S^g \mathcal{X}} \simeq \bigwedge_{\mathcal{O}_{S^g \mathcal{X}}} (\mathcal{M}^{S_g})$ (17), a morphism $\varphi: \mathcal{S} \rightarrow S^g \mathcal{X}$ is characterized by the induced morphism $\bar{\varphi}: \mathcal{S} \rightarrow S^g X$ and a morphism $\psi: \mathcal{M}^{S_g} \rightarrow \mathcal{O}_{\mathcal{S}}$ of $\mathcal{O}_{S^g \mathcal{X}}$ -modules, where $\mathcal{O}_{\mathcal{S}}$ has the $\mathcal{O}_{S^g \mathcal{X}}$ -module given by $\bar{\varphi}$. This will be very useful in the next section.

Our next step is to prove the main result of this paper, the *Representability Theorem for the Supersymmetric product*.

Theorem 4. *The natural morphism of functors:*

$$\begin{aligned} \Theta: \text{Hom}(\mathcal{S}, S^g \mathcal{X}) &\rightarrow \text{Div}_S^g(\mathcal{X} \times \mathcal{S}) \\ \varphi &\mapsto (1 \times \varphi)^{-1}(\mathcal{Z}^g) \end{aligned} \quad (32)$$

is a functorial isomorphism between the functor of points of the supersymmetric product $S^g \mathcal{X}$ and the functor of relative superdivisors of degree g .

Proof. Let us outline a sketch of the proof. The first question is the construction of the inverse map of Θ . Let $\mathcal{Z}$ be a relative superdivisor of degree g of $\mathcal{X} \times \mathcal{S} \rightarrow \mathcal{S}$ defined by an ideal J of $\mathcal{O}_{\mathcal{X} \times \mathcal{S}}$. We need a morphism $\varphi: \mathcal{S} \rightarrow S^g \mathcal{X}$ such that $(1 \times \varphi)^{-1}(\mathcal{Z}^g) = \mathcal{Z}$. However, as we have already seen, this is equivalent to giving a morphism $\bar{\varphi}: \mathcal{S} \rightarrow S^g X$ and a morphism $\psi: \mathcal{M}^{S_g} \rightarrow \mathcal{O}_{\mathcal{S}}$ of $\mathcal{O}_{S^g \mathcal{X}}$ -modules.

The morphism $\bar{\varphi}: \mathcal{S} \rightarrow S^g X$ is what one obtains through Theorem 2 from the reduction $\bar{\mathcal{Z}}$, which is an ordinary relative divisor of degree g of $X \times \mathcal{S} \rightarrow \mathcal{S}$.

The morphism of $\mathcal{O}_{S^g \mathcal{X}}$ -modules $\psi: \mathcal{M}^{S_g} \rightarrow \mathcal{O}_{\mathcal{S}}$ can now be defined locally. We can then take conformal coordinates (z, θ) on $\mathcal{X}$, such that the ideal J of $\mathcal{Z}$ is generated by an element:

$$f = z^g - (a_1 + \theta b_1)z^{g-1} + \dots + (-1)^g(\bar{a}_g + \theta b_g) \quad (33)$$

where the a_i 's are even and the b_j 's are odd elements in $\mathcal{O}_S$. Since the module $\mathcal{M}^{S_g}$ is generated by the odd symmetric functions ς_i , we can define a morphism $\psi: \mathcal{M}^{S_g} \rightarrow \mathcal{O}_S$ of $\mathcal{O}_{S^g X}$ -modules by imposing the condition that $\varsigma_i \mapsto b_i$ for $i = 1, \dots, g$.

On the other hand, since the ring morphism induced by $\bar{\varphi}: S \rightarrow S^g X$ maps the symmetric functions s_i to the coefficients a_i of the local generator f of J (see (23)), in our coordinate open subset we have $(1 \times \varphi)^{-1}(Z^s) = Z$.

Using the conformal structure and, more specifically, the remark following Theorem 3, we can prove that this construction of ψ is independent of the choice of the conformal coordinates, thus providing the global morphism we are seeking. In this way, we have a morphism of functors:

$$\Xi: \text{Div}_S^g(\mathcal{X} \times S) \rightarrow \text{Hom}(S, S^g \mathcal{X}). \quad (34)$$

From the above local description we can easily see that Ξ and Θ are one the inverse of the other. $\square$

Remark. As in the non-graded case, the meaning of Theorem 4 is that the supersymmetric product $S^g \mathcal{X}$ of a SUSY-curve and its universal superdivisor Z^s parametrize relative superdivisors of degree g for every SUSY-family $\mathcal{X} \times S \rightarrow S$, i.e., every relative superdivisor can be obtained in a unique way as a pullback of the universal relative superdivisor of the supersymmetric product.

According to the remark following Theorem 3, the odd coefficients of the equation of a relative superdivisor can be obtained 'generically' by derivation of the even ones with the aid of the conformal structure. Such a statement, which is certainly meaningless for a particular divisor because we cannot derive its coefficients (which are constant) is nevertheless behind our definition of relative superdivisor.

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